The maximum time of 2-neighbour bootstrap percolation in grid graphs and parametrized results

Thiago Marcilon
Universidade Federal do Ceará (UFC)
Fortaleza, Brazil

This is a joint work with
Rudini Sampaio (UFC, Fortaleza, Brazil)

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2-Neighbour Bootstrap Percolation

2-Neighbour Bootstrap Percolation $\Leftrightarrow$ Infection $\Leftrightarrow$ P_3 convexity

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- Initially infected set $S = S_0 \subseteq V(G)$
- Spreading Rule:
  $S_{i+1} = S_i \cup \{\text{all vert. having 2 infected neighbours}\}$

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Some Definitions/Notations

- S is a percolating set of $G \Leftrightarrow \exists k\ S_k = V(G)$
- Let $t_S(G)$ be the smallest value k to which $S_k = V(G)$
- Let $t(G) = \max_{S \in Ps(G)} t_S(G)$, where $Ps(G)$ is the set of all percolating sets of G

Motivation

It models:

- Fluid flow [Adler & Aharony, 1988, Journal of Physics A 21]
- Sandpile growth [Fey et al., 2010, Journal of Statistical Physics 138]
- Living neural networks [Amini, 2010, J. of Statistical Physics 141]
- Failure in storage arrays [Kirkpatrick et al., 2002, Physica A 314]
- Opinion diffusion [Dreyer et al., 2009, Discrete Applied Math. 157]

Example of Infection

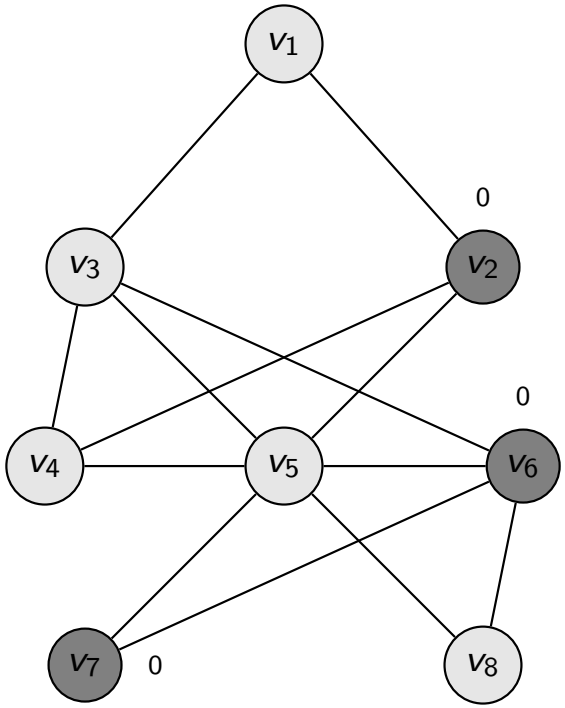

Figure 1: Spreading of the set $S = \{v_2, v_6, v_7\}$.

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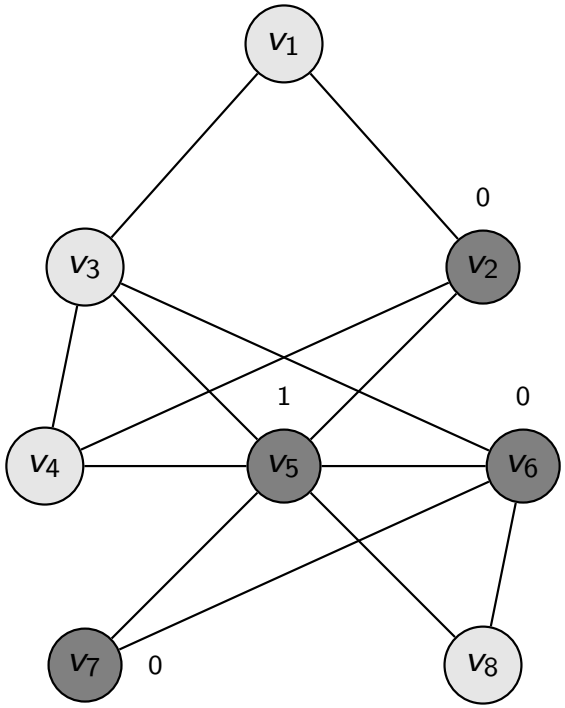

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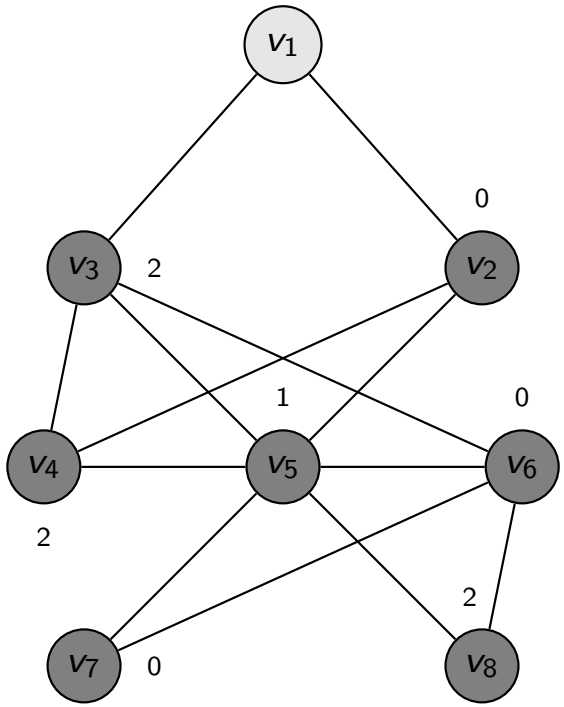

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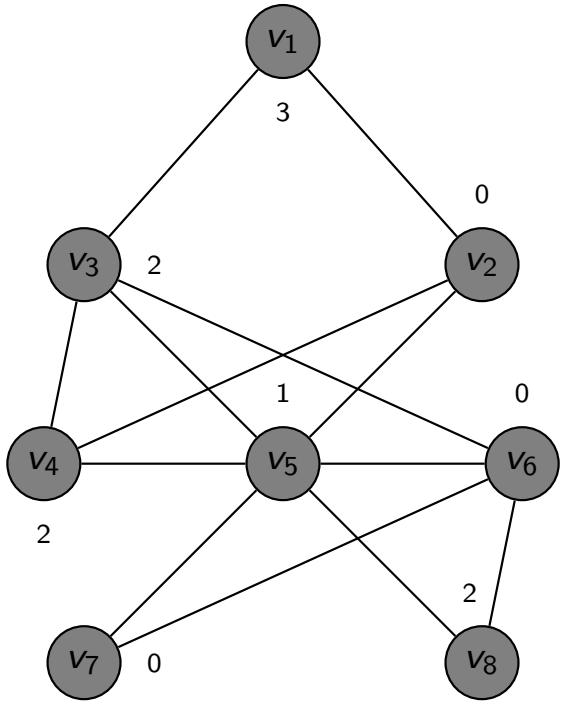

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The Percolation (Infection) Time Problem

$$t(G) \geq k \ ?$$

Example

$t(G) \geq 4$?

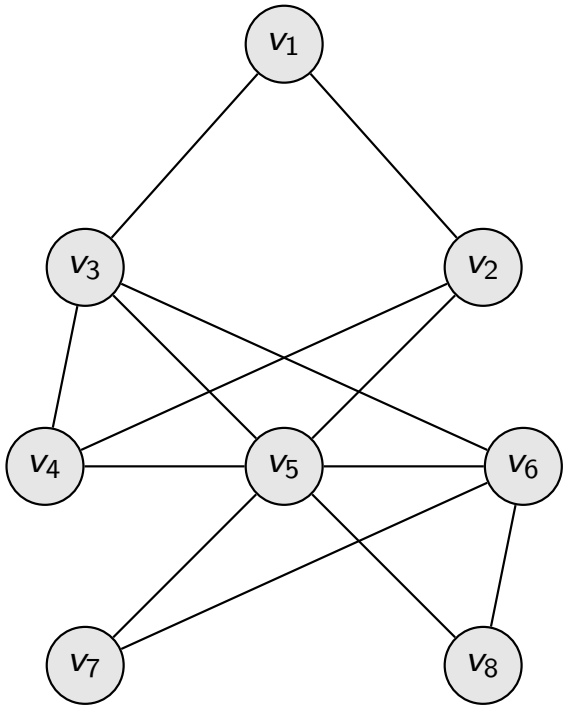

Figure 2: Graph G.

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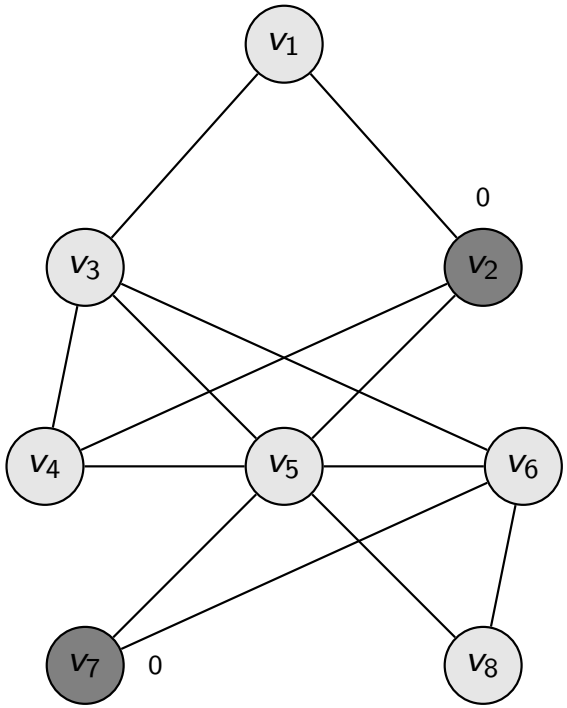

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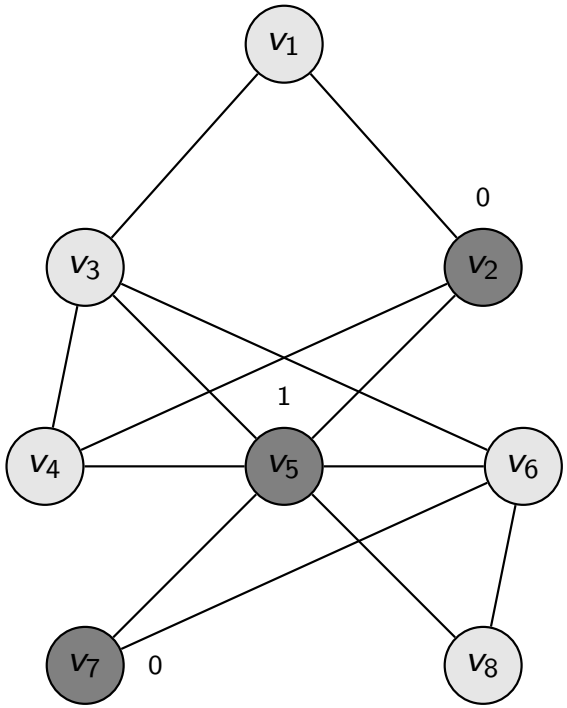

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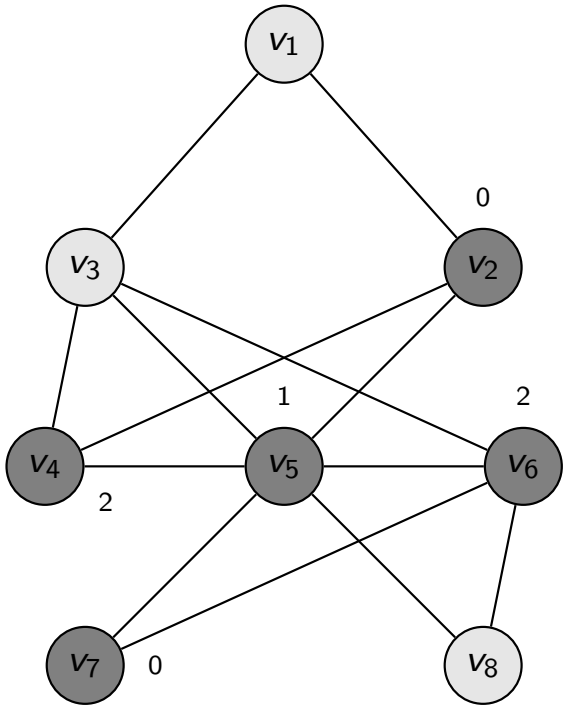

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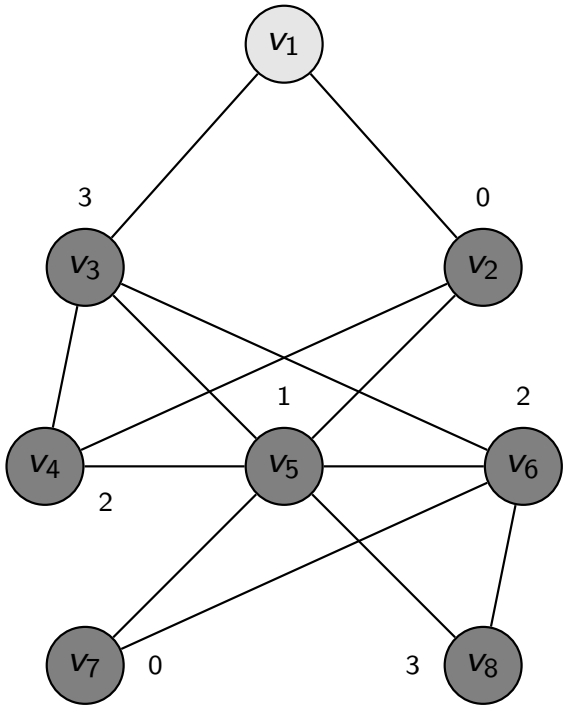

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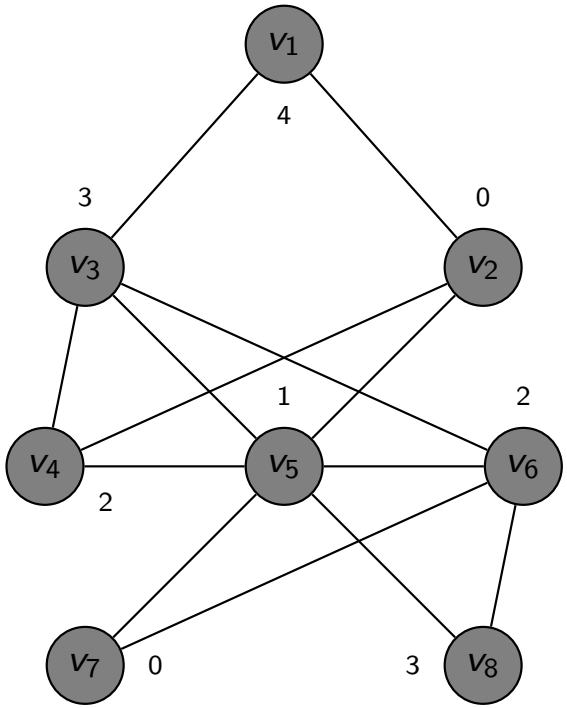

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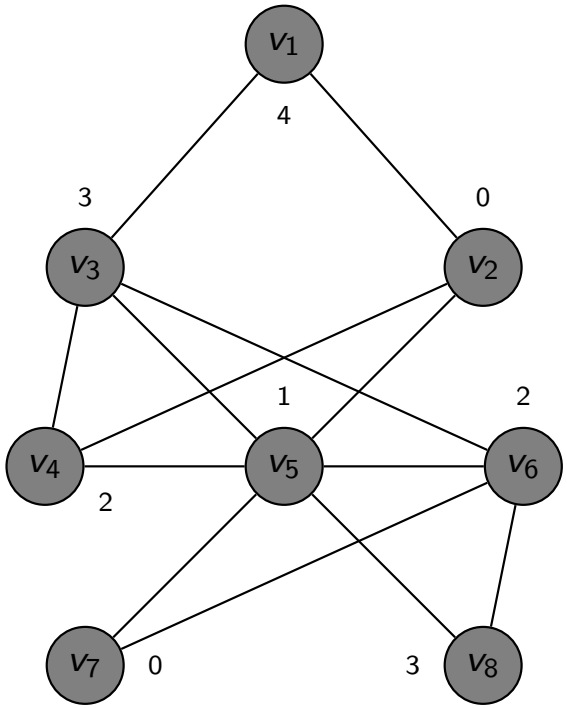

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Results for fixed K [Marcilon et al., 2014, WG]

- Polynomial for any fixed $k \leq 3$
- NP-Complete for any fixed $k \geq 4$
- Polynomial when G is bipartite for any fixed $k \leq 4$
- NP-Complete when G is bipartite for any fixed $k \geq 5$

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Percolation Time Problem in grid graphs

Percolation Time Problem in grid graphs with maximum degree 3 is NP-complete by reduction from Longest Path problem in grid graphs with maximum degree 3:

- Each edge from the original graph is subdivided twice
- Add some neighbors to the new vertices so each 4x4 block should look like the Figure 3

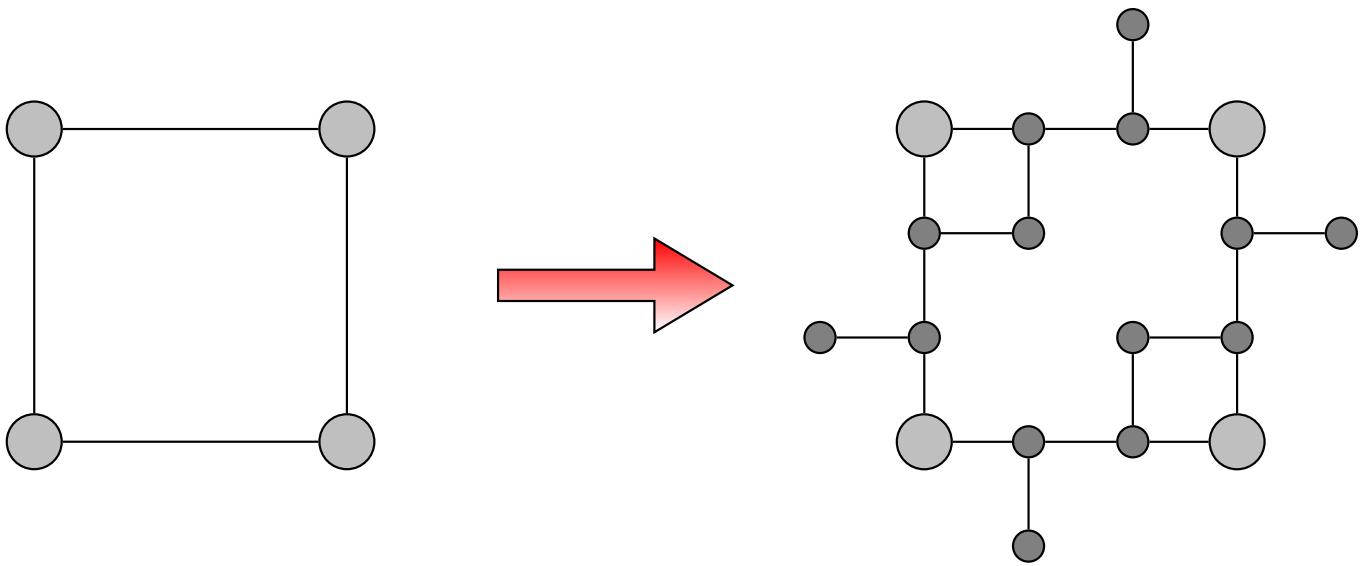

Figure 3: Original 2x2 block to the left and 4x4 block after the addition of the neighbors to the right

Percolation Time problem in grid graphs

Theorem

G has a path with size greater or equal to $k \Leftrightarrow t(G') \geq 3k + 2$.

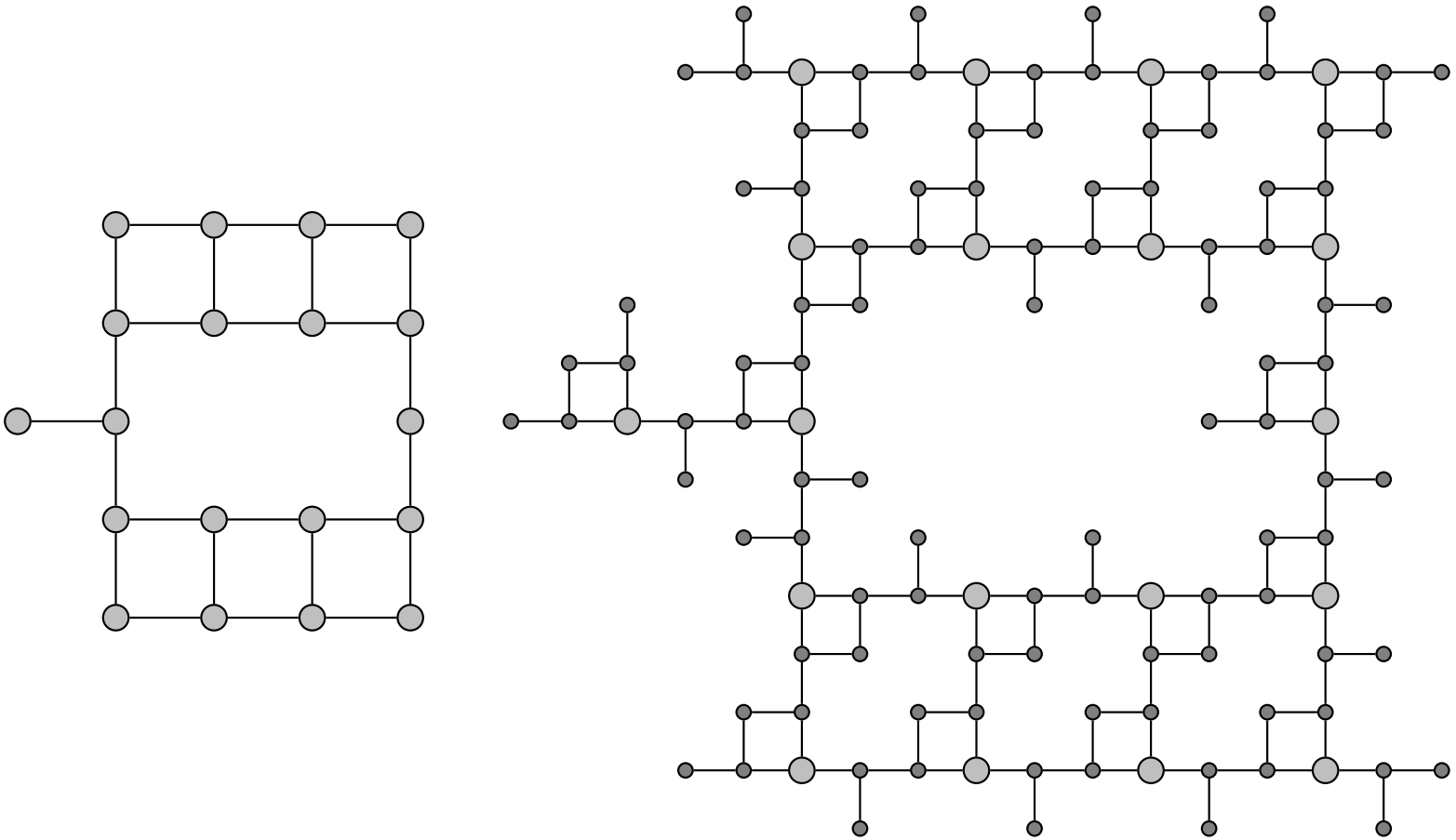

Figure 4: A grid graph G to the left and the graph G' resulting from the reduction applied to G to the right.

Future Work

Complexity of the Percolation Time Problem in:

- Solid grid graphs (without any restriction on the maximum degree)
- Graphs with fixed treewidth
- Graphs with fixed cliquewidth