The Maximum Time of 2-Neighbour Bootstrap Percolation

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This is a joint work with
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2-Neighbour Bootstrap Percolation

Percolation Time Problem

Our Results

Conclusion

2-Neighbour Bootstrap Percolation

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Infection on Graphs

- Initial infected set $S = S_0 \subseteq V(G)$
- Spreading Rule:
  $S_{i+1} = S_i \cup \{\text{all vert. having 2 infected neighbours}\}$

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Some Definitions/Notations

- S is a percolating set of $G \Leftrightarrow \exists k\ S_k = V(G)$
- Let $t_S(G)$ be the smallest value k to which $S_k = V(G)$
- Let $t(G) = \max_s t_S(G)$

Example of Infection

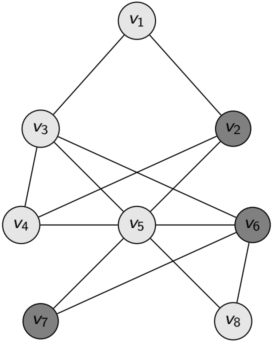

Figure 1: Spreading of the set $S = \{v_2, v_6, v_7\}$.

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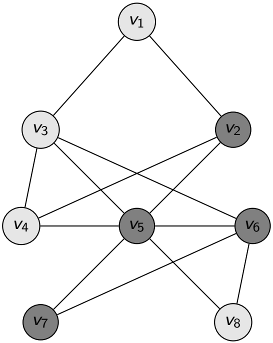

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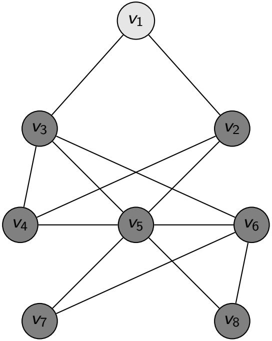

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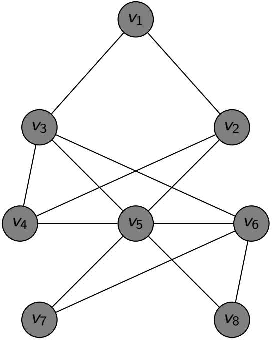

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The Percolation Time Problem

$$t(G) \geq k \ ?$$

Example

$$t(G) \geq 4 \ ?$$

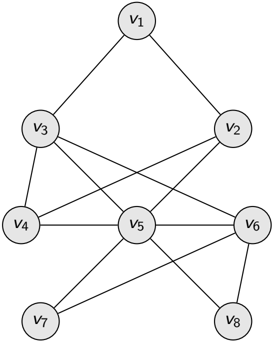

Figure 2: The Graph G.

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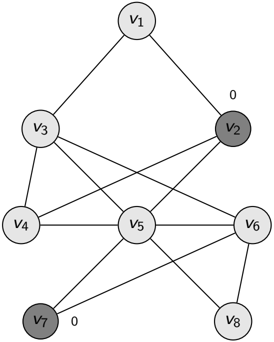

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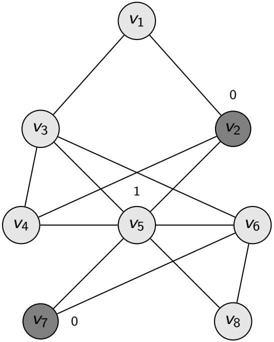

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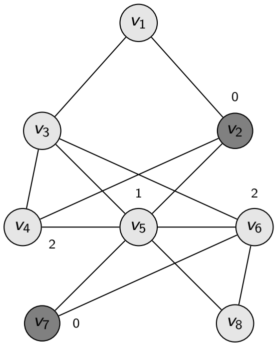

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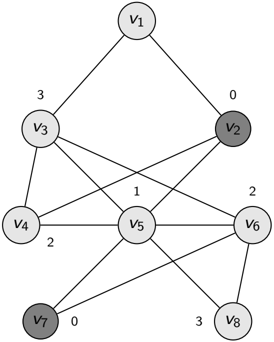

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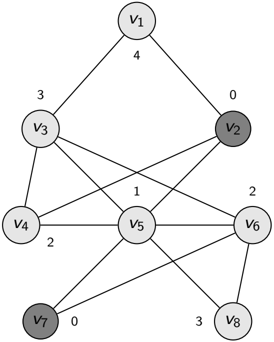

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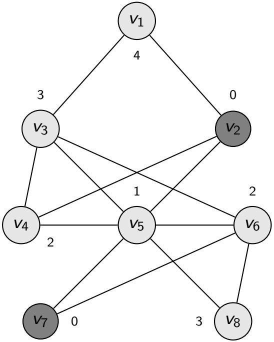

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Previous Results [Benevides et al., 2013]

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NP-complete Results

- General graphs
- Bipartite graphs
- Planar graphs

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Polynomial Results

- Trees (Linear Time)
- Chordal graphs

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Polynomial Results

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Results for fixed K

- $t(G) \geq 2$ is polynomial
- $t(G) \geq 4$ is NP-Complete
- $t(G) \geq 7$ is NP-Complete for G bipartite

Our Results

Results for Bipartite Graphs

Results for General Graphs

Our Results

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- $t(G) \geq 3$ is polynomial (mn^3)

Results for General Graphs

Our Results

Results for Bipartite Graphs

- $t(G) \geq 3$ is polynomial (mn^3)
- $t(G) \geq 4$ is polynomial (mn^{13})

Results for General Graphs

Our Results

Results for Bipartite Graphs

- $t(G) \geq 3$ is polynomial (mn^3)
- $t(G) \geq 4$ is polynomial (mn^{13})
- $t(G) \geq 5$ is NP-Complete

Results for General Graphs

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Results for General Graphs

- $t(G) \geq 3$ is polynomial (mn^5)

$t(G) \geq 5$ is NP-Complete (for G bipartite)

Reduction from the problem **3SAT**:

- For each clause C_i, add the gadget in the Figure 3
- For each pair of literals $l_{i,a}$ and $l_{j,b}$, add a vertex $y_{(i,a),(j,b)}$ and link this vertex to the vertices $w_{i,a}$ and $w_{j,b}$
- Add a vertex z and link it to all vertices $y_{(i,a),(j,b)}$ and add a vertex adjacent only to z.

$t(G) \geq 5$ Reduction from **3SAT**

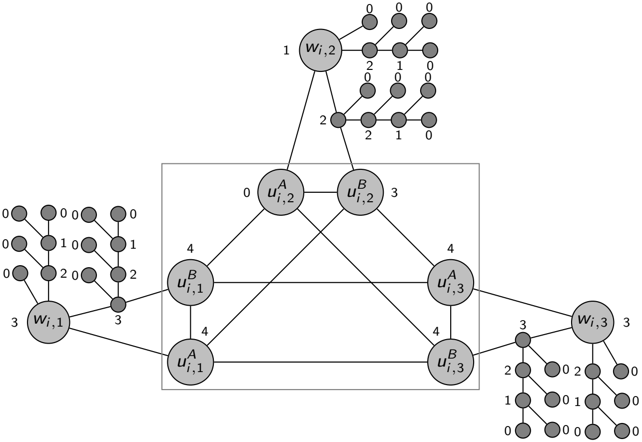

Figure 3: Bipartite gadget for each clause C_i

$t(G) \geq 5$ Reduction from **3SAT**

3SAT Instance: $(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee x_3)$

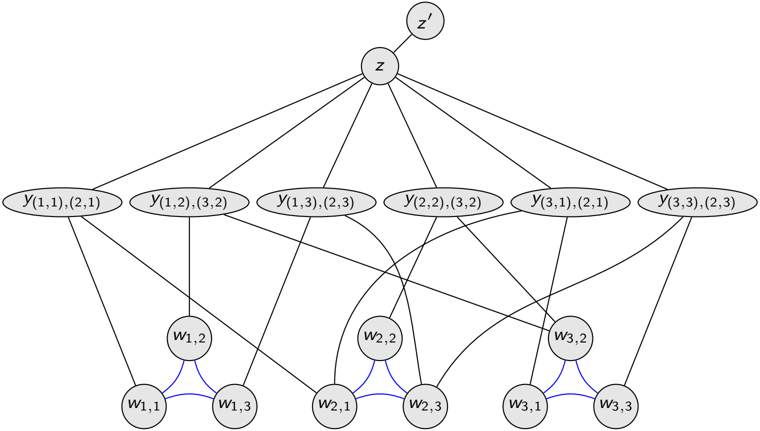

Figure 4: Graph resulting from the reduction from an instance of **3SAT**

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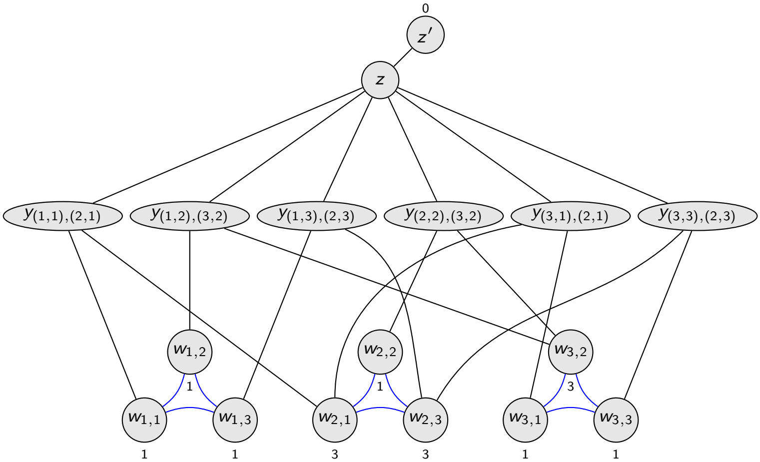

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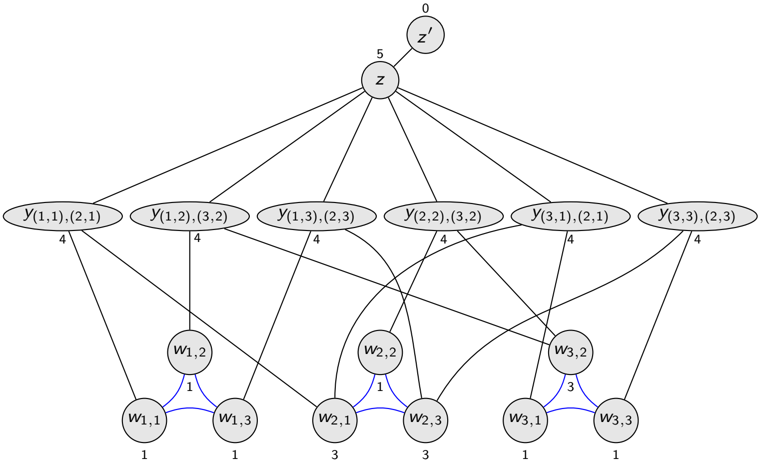

Figure 4: Graph resulting from the reduction from an instance of **3SAT**

$t(G) \geq 3$ is Polynomial (for G bipartite)

Some Definitions

- $\mathbf{T_0} = \{v \in V(G) | \text{v has degree } 1\}$
- $\mathbf{N_i(u)} = \{v \in V(G) | \text{dist. between } u \text{ and } v \text{ is } i\}$
- $\mathbf{N_{\geq i}(u)} = \{v \in V(G) | \text{dist. between } u \text{ and } v \text{ is } \geq i\}$

Theorem

$t(G) \geq 3$ iff there are vertices $u \in V(G), v \in N(u), s \in N_2(u)$ s.t. $\{v, s\} \cup N_{\geq 3}(u) \cup T_0$ percolates u at time 3.

Corollary

There is an algorithm that solves in bipartite graphs the Percolation Time Problem for a fixed $k = 3$ in time mn^3.

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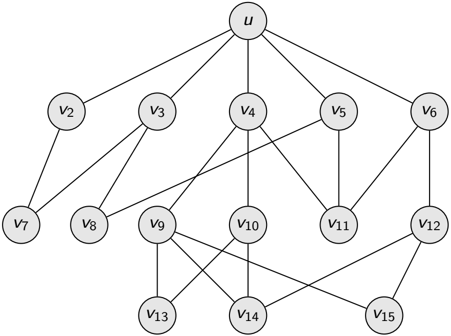

Figure 5: Graph G

$t(G) \geq 3$ is Polynomial (for G bipartite)

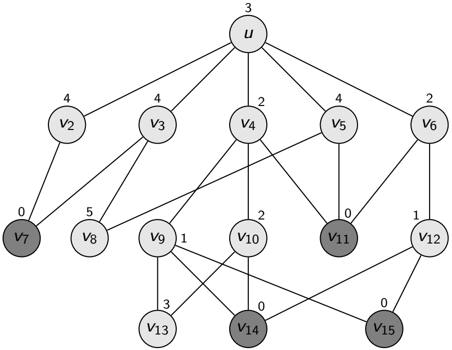

Figure 5: Vertices' infection time when G is infected by the percolating set $S = \{v_7, v_{11}, v_{14}, v_{15}\}$

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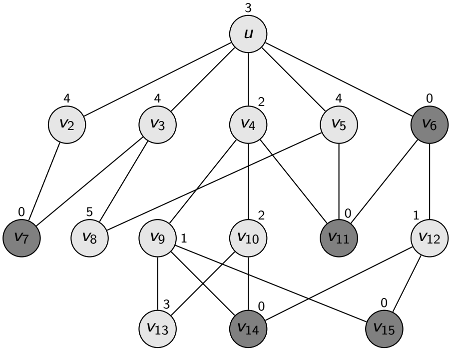

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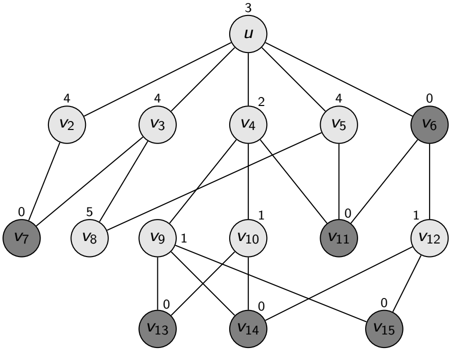

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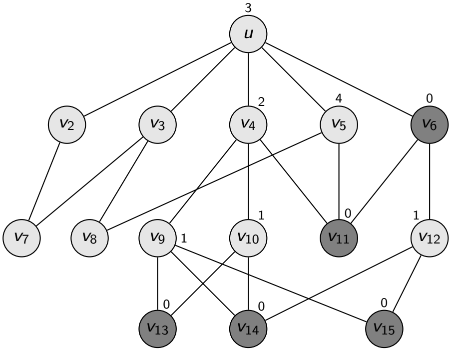

Figure 5: Vertex u is infected at time 3 by the set $\{v_6, v_{11}, v_{13}, v_{14}, v_{15}\}$

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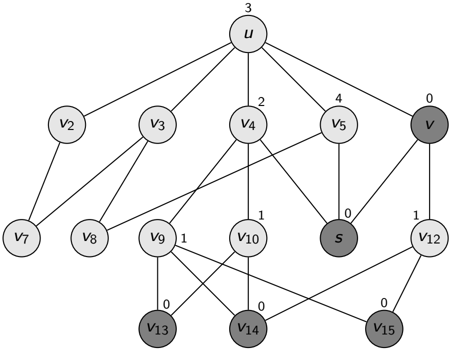

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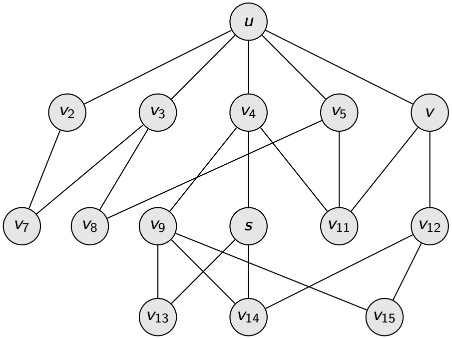

Let $S := \{v, s\} \cup N_{\geq 3}(u) \cup T_0$

Update vertices' infection time

While S does not infect all vertices

Choose any $x \in N_2(u)$ not infected

and let $S := S \cup \{x\}$

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Figure 6: Building a percolating set S from $\{v, s\} \cup N_{\geq 3}(u) \cup T_0$.

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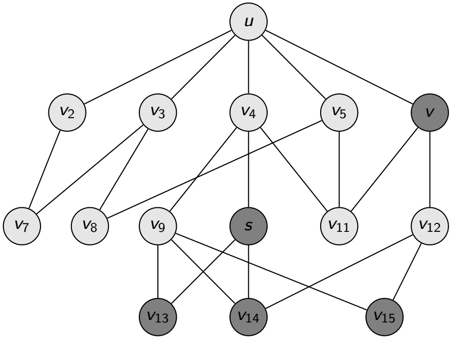

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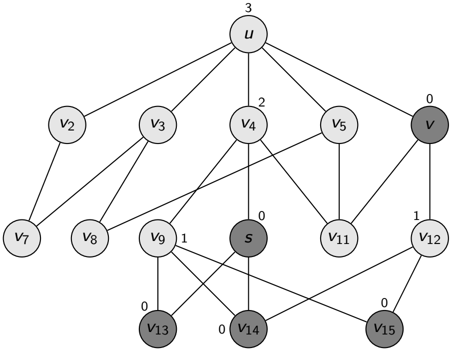

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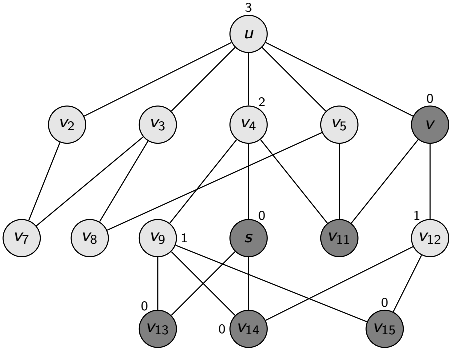

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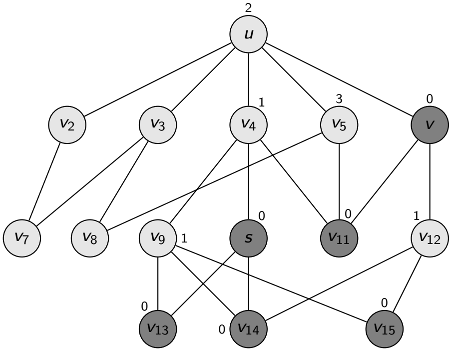

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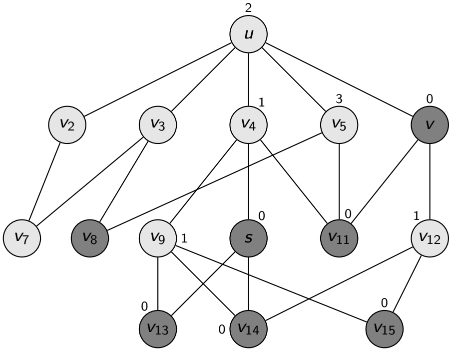

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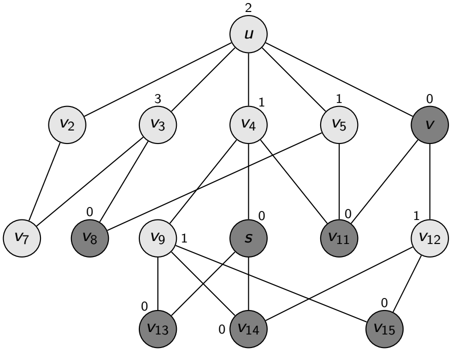

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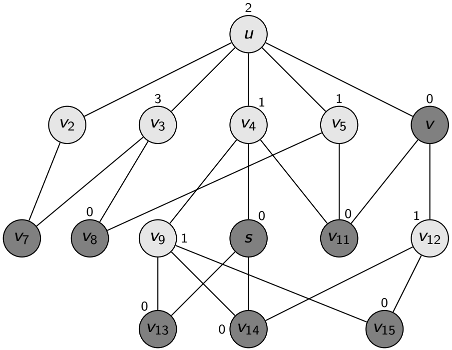

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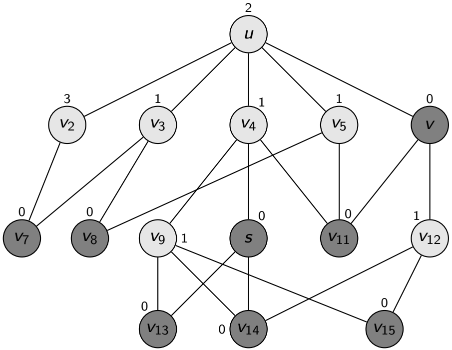

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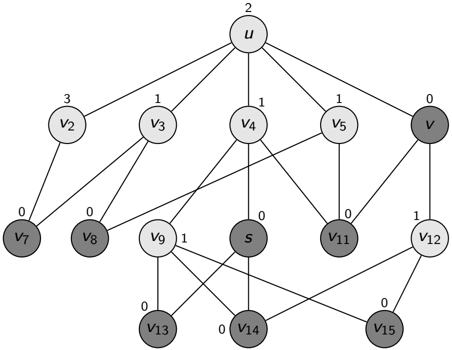

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Figure 6: Percolating set $S = \{v, s, v_{11}, v_8, v_7\} \cup N_{\geq 3}(u) \cup T_0$

$t(G) \geq 3$ is Polynomial (for any graph G)

Theorem

$t(G) \geq 3$ iff there are $u \in V(G)$, $T_0 \in \mathcal{T}_0^u$, $k \leq 4$ and a set $F \subseteq \binom{V(G)}{k}$ s.t. $T_0 \cup N_{\geq 3}(u) \cup F$ percolates u at time 3 then .

Corollary

There is an algorithm that solves the Percolation Time Problem for a fixed $k = 3$ in time mn^5.

$t(G) \geq 4$ is Polynomial (for G bipartite)

Theorem

$t(G) \geq 4$ iff there are $u \in V(G), T_0 \in \mathcal{T}_0^u, k \leq 10, F \subseteq \binom{V(G)}{k}$ s.t. $T_0 \cup F$ percolates some vertex at time 4.

Corollary

There is an algorithm that solves in bipartite graphs the Percolation Time Problem for a fixed $k = 4$ in time mn^{13}.

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Main Contributions

- We closed the gap between polynomial time and NP-Complete problems
- We obtained structural characterizations for the graphs in the polynomial problems

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Future Work

Percolation Time Problem in:

- Restricted degree graphs
- Subgraph and induced subgraph of grids