Inapproximability results for graph convexity parameters

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This is a joint work with
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Contents

Convexity in Graphs

- Given a graph G, a family C of subsets of $V(G)$ is a **convexity on** G if
  - $\emptyset, V(G) \in C$
  - C is closed under intersection
- Every member of C is a **convex set**
- The **convex hull of** S is the smallest convex set containing S, denoted by $hull(S)$.

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Convexity in Graphs

Convexities (P_3, geodetic, monophonic, m^3)

- P_3-convexity - all paths with 3 vertices
- Geodetic convexity - all shortest paths
- Monophonic convexity - all induced paths
- m^3-convexity - all induced paths of size at least 3

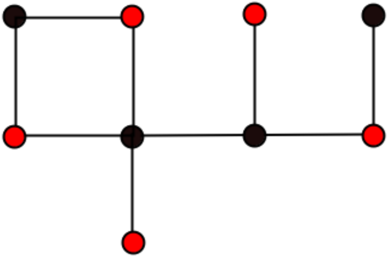
Convexity in Graphs

Intervals (P_3, geodetic, monophonic, m^3)

Given a graph G and a set S of vertices, let:

- $I_{P_3}(S) \ = \ S \cup \{P_3\text{'s between vert. of } S\}$
- $I_{geo}(S) \ = \ S \cup \{\text{minimum paths between vert. of } S\}$
- $I_{mo}(S) \ = \ S \cup \{\text{induced paths between vert. of } S\}$
- $I_{m^3}(S) \ = \ S \cup \{\text{induced paths length} \geq 3 \text{ between vert. of } S\}$

P_3-Convex hull



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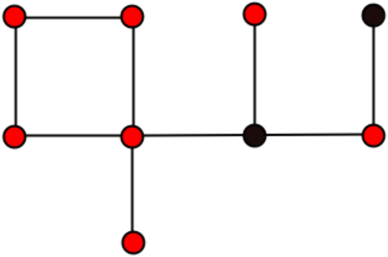

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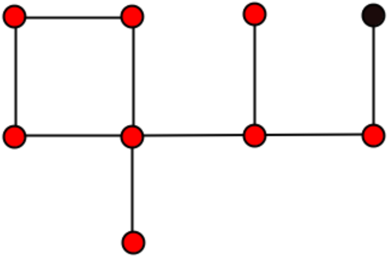

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Some definitions

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- Radon number later ...

Complexity results

Geodetic convexity (NP-hard results)

- Hull number [Araújo et al., 2013, TCS]
- Interval number [Dourado et al., 2010, DM]
- Convexity number [Dourado et al., 2012, G&C]
- Carathéodory number [Dourado et al., 2013, sub]

P_3 convexity (NP-hard results) (bipartite graphs)

- Hull number [Centeno et al., 2011, TCS]
- Interval number [Centeno et al., 2009, ENDM]
- Convexity number [Centeno et al., 2009, ENDM]
- Carathéodory number [Barbosa et al., 2012, SIAM J.DM]

Our results

P_3-hull number is APX-hard in bipartite graphs

L-Reduction from MAX-2-SAT-3 (APX-Complete)

- Every clause has at most 2 literals,
- Every literal is in some clause and,
- For every variable x_i, there are at most 3 clauses containing either x_i or $\overline{x_i}$.

Example

$\Phi = (x_1 \vee x_2) \wedge (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2)$

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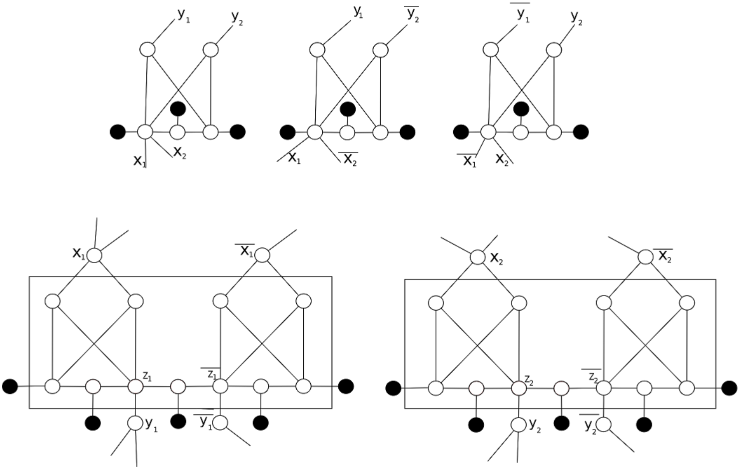

Figura: Max-2-Sat-3 formula: $\Phi = (x_1 \vee x_2) \wedge (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2)$

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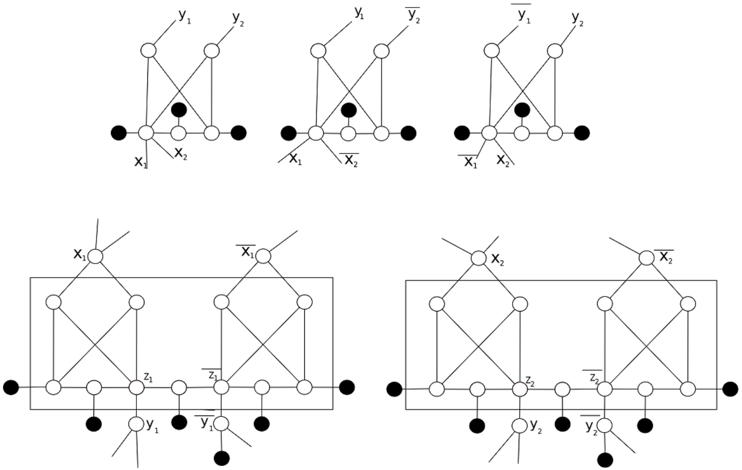

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- Every hull set contains
  - all black vertices $= b$
  - one vertex for every variable gadget
- hull number is at least $b + k + 3m$

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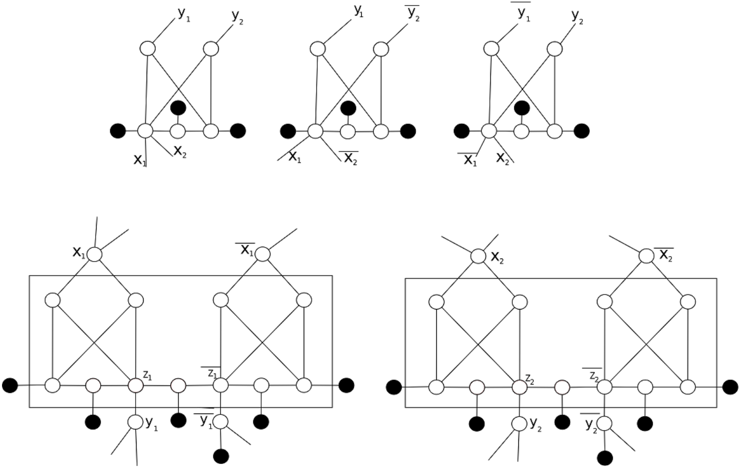

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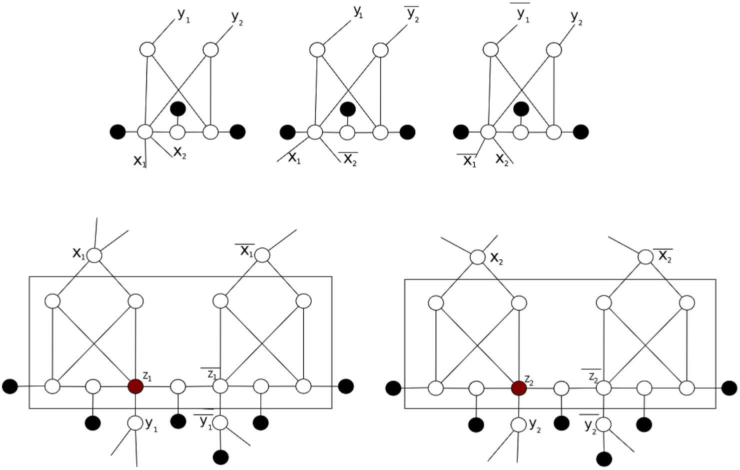

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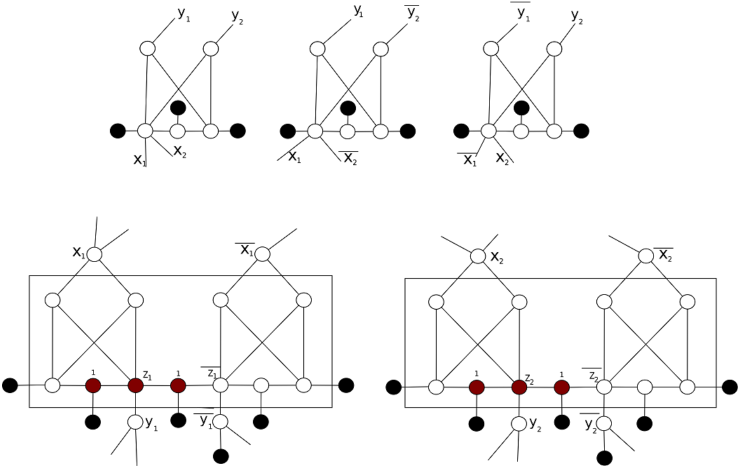

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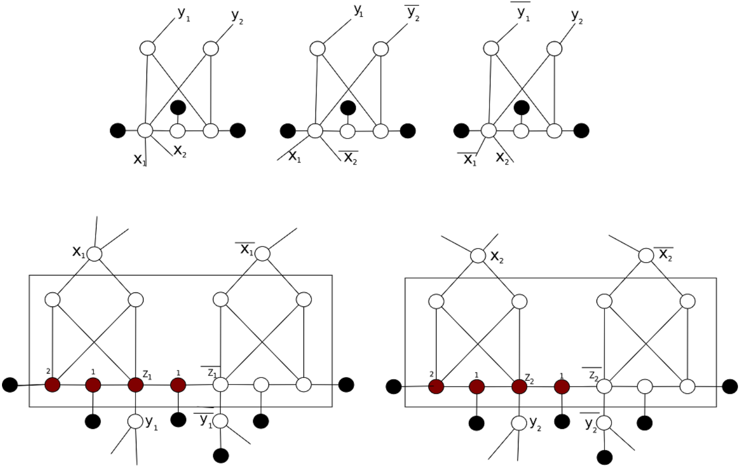

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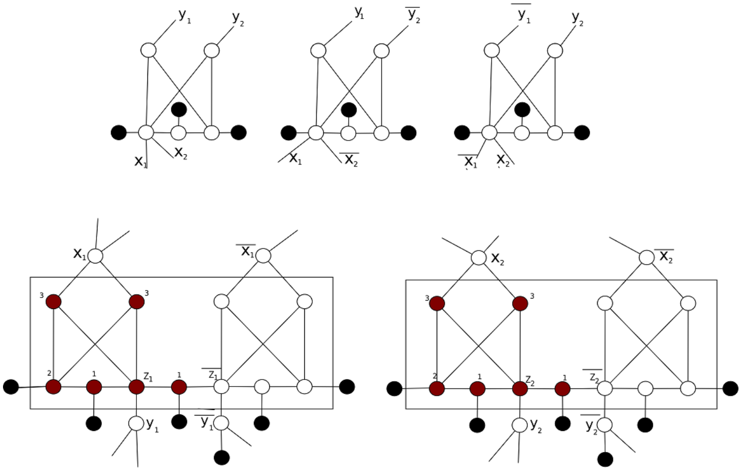

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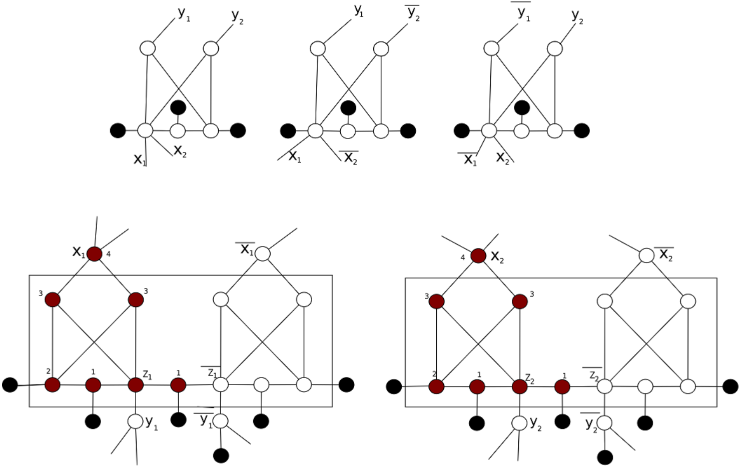

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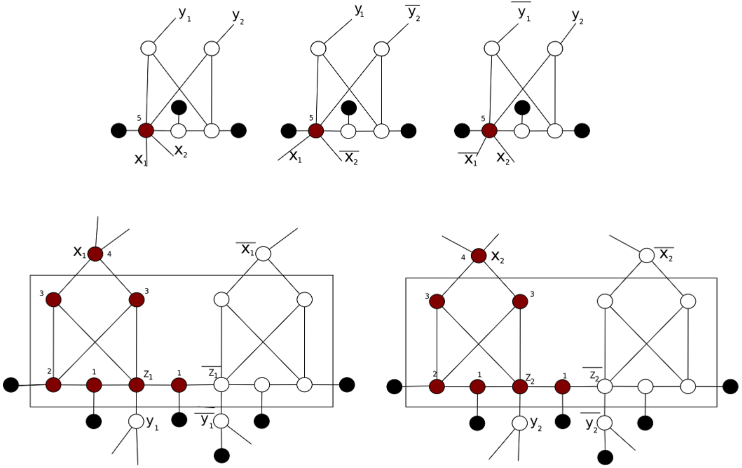

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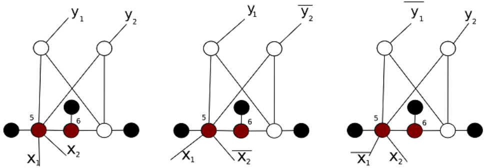

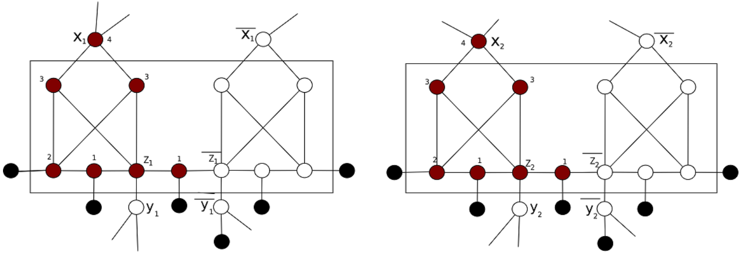

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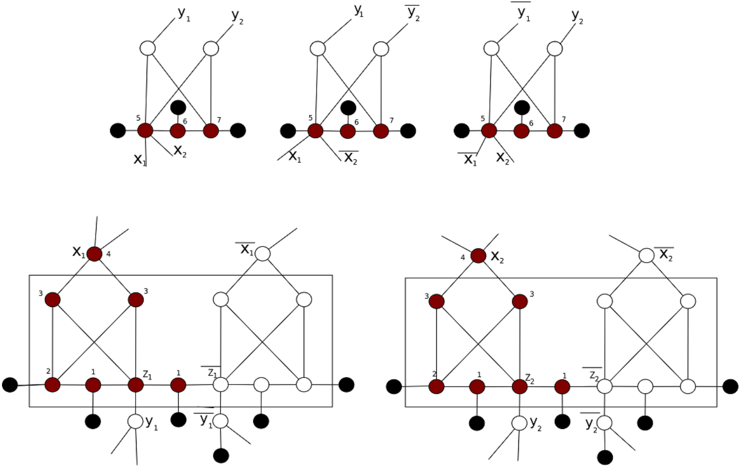

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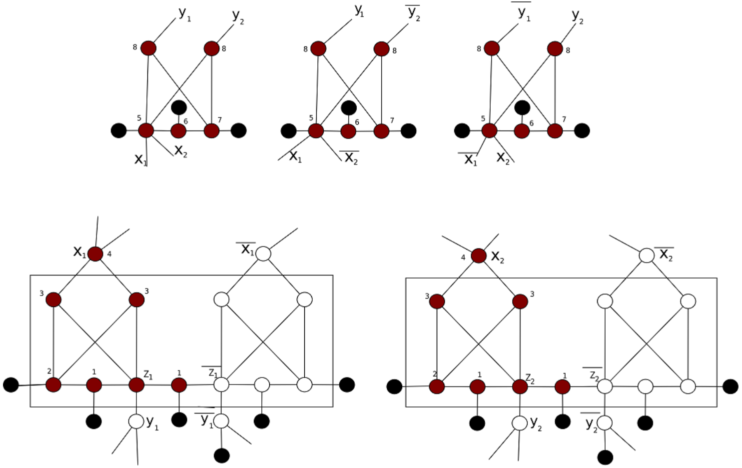

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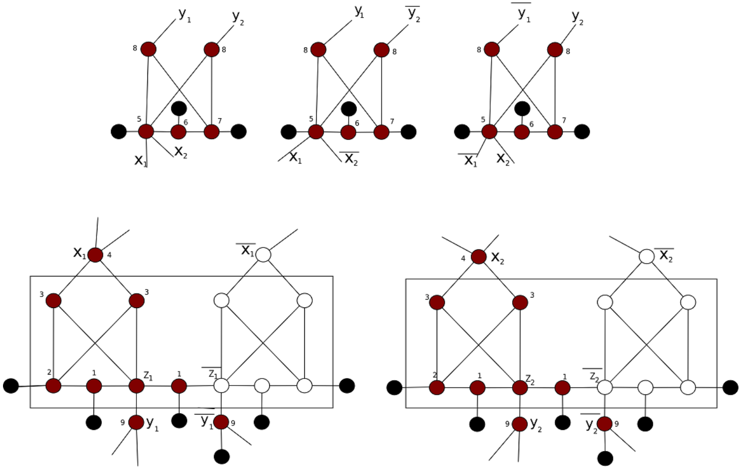

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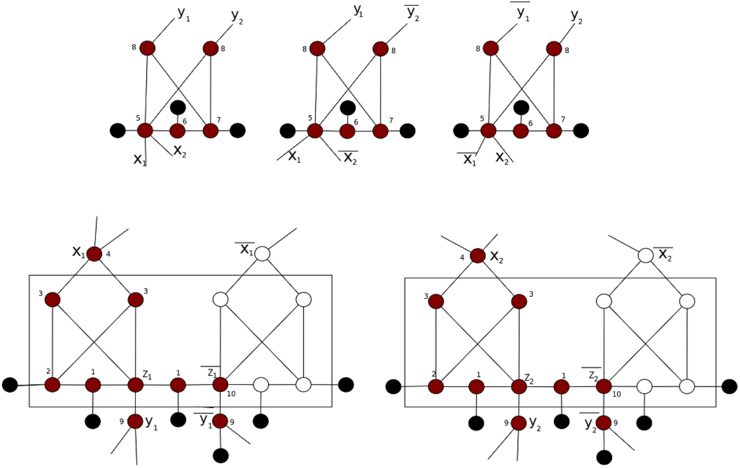

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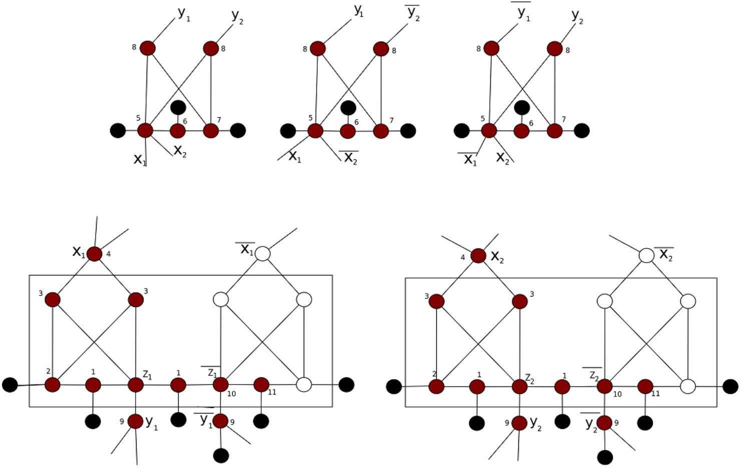

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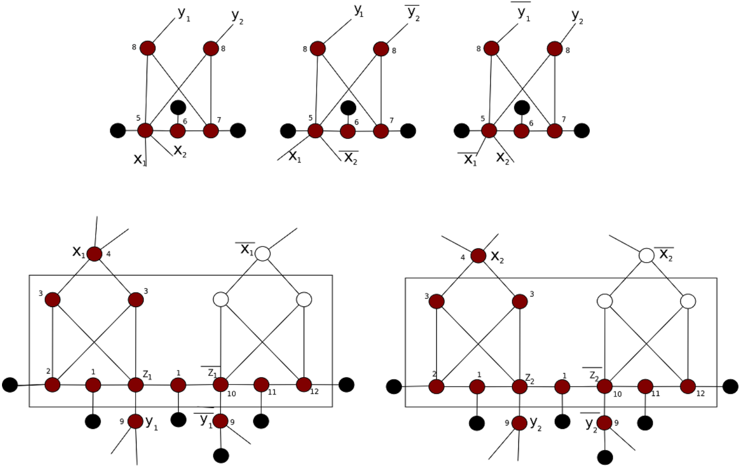

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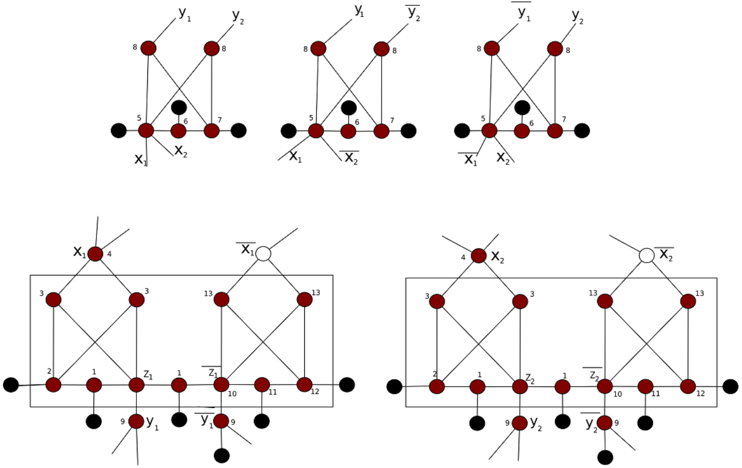

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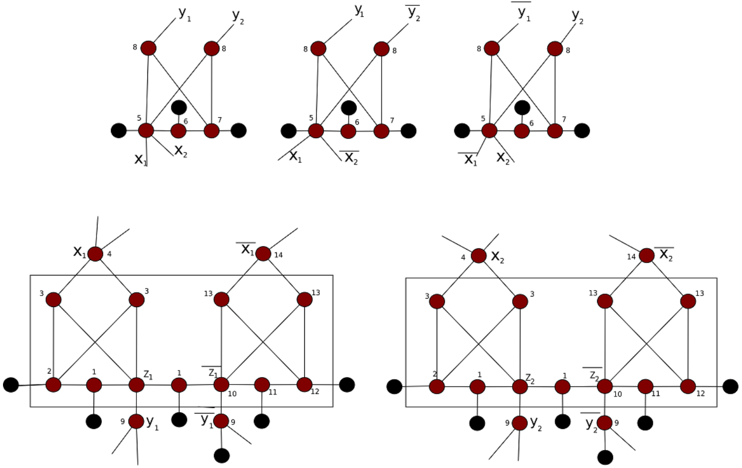

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Innaproximability of P_3-convexity number

$O(n^{1-\varepsilon})$-innaproximable in bipartite graphs in polynomial time unless P=NP.

Reduction from SET-PACKING ($O(n^{1-\varepsilon})$-innaproximable problem)

Given m sets $S_1, \ldots, S_m$, determine the maximum k s.t. there exist k pairwise disjoint sets.

Example ($k = 3$, $m = 5$)

- $S_1 = \{a, b, c\}$, $S_2 = \{b, f, g\}$
- $S_3 = \{d, e, f\}$, $S_4 = \{c, e, g\}$
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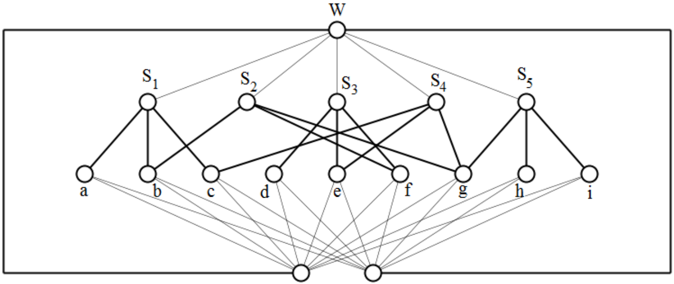

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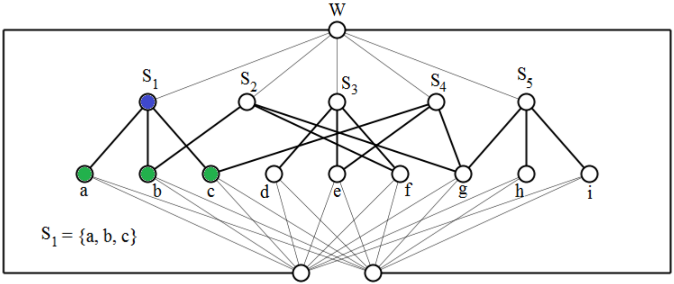

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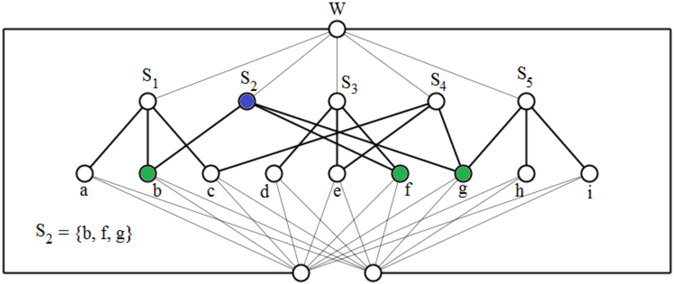

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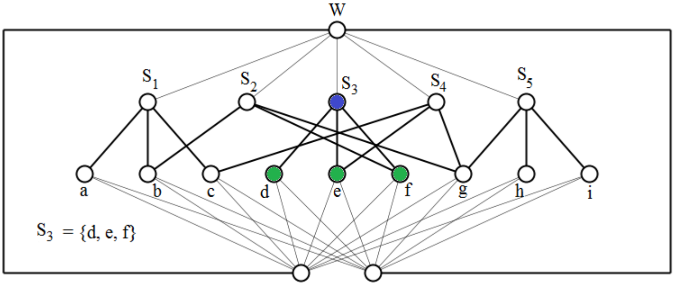

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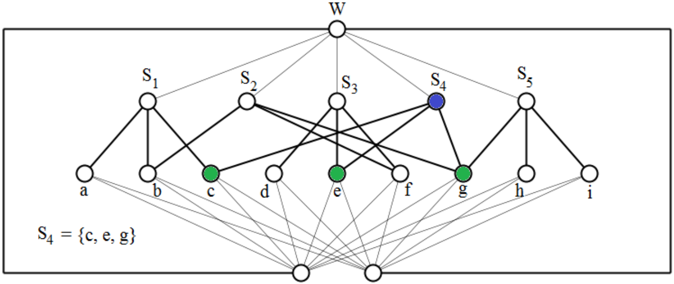

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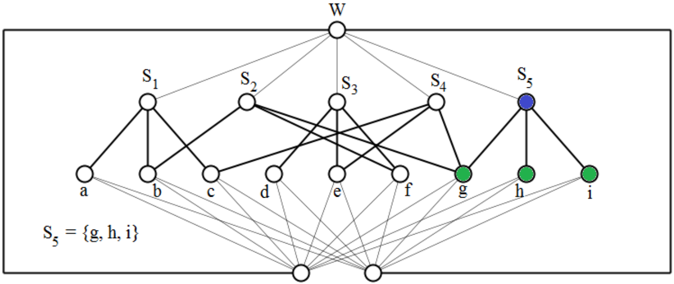

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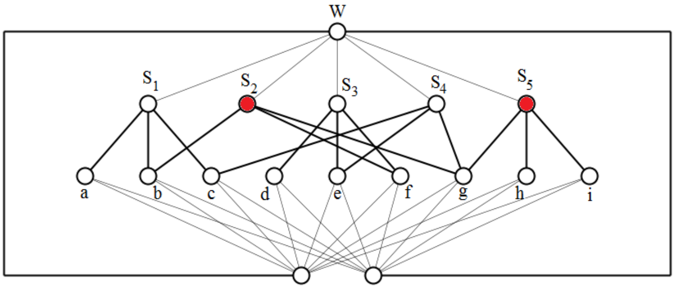

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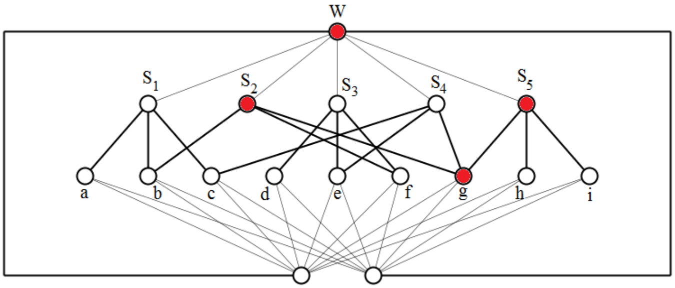

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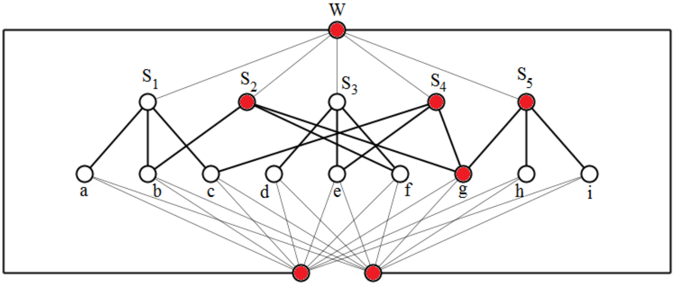

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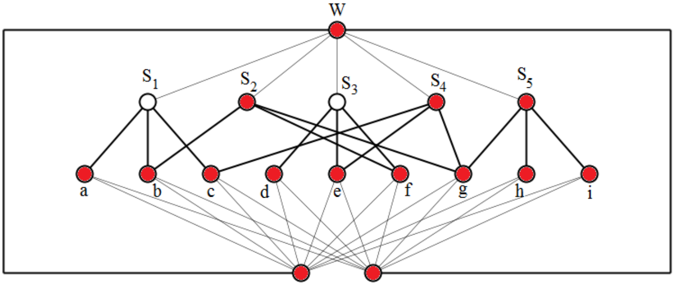

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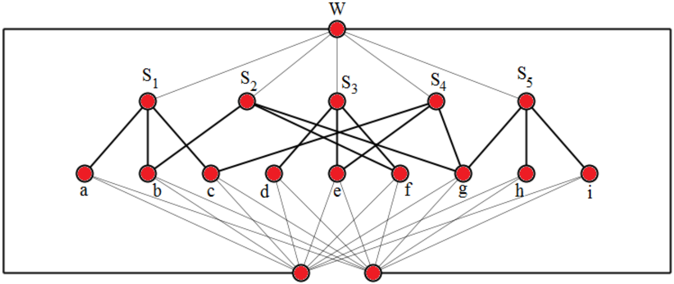

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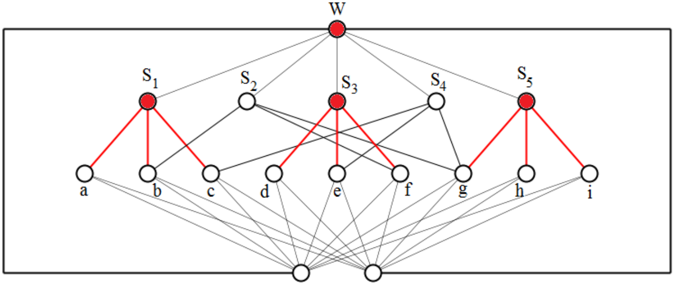

Figura: Reduction from SET-PACKING

Innaproximability of P_3-Radon number

- A partition $S_1 \cup S_2$ of a set S is a **Radon partition of** S if $hull(S_1) \cap hull(S_2) \neq \emptyset$
- The **Radon number** of a graph G is the smallest k for which every $S \subset V(G)$, with size at least k, admits a Radon partition

Innaproximability of P_3-Radon number

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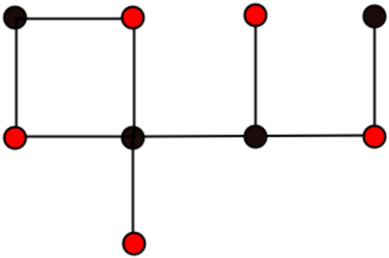
P_3-Radon partition



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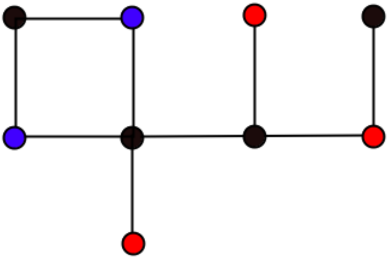

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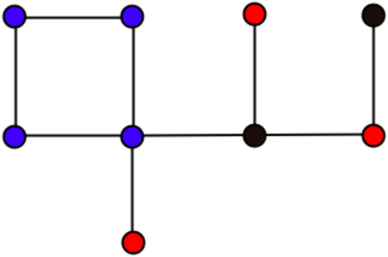

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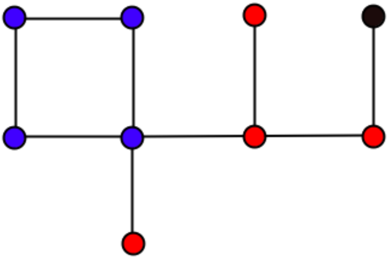

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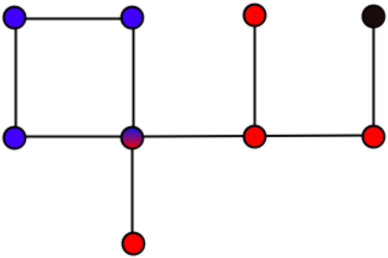

Innaproximability of P_3-Radon number

- The P_3-Radon number problem is $O(n^{1-\varepsilon})$-innaproximable in bipartite graphs in polynomial time unless P=NP.
- From SET-PACKING problem

Innaproximability of P_3-interval number

- The P_3-interval number problem is $O(\log n)$-innaproximable in bipartite graphs in polynomial time unless P=NP.
- From SET COVER problem

Innaproximability of P_3-Carathéodory number

- P_3-Carathéodory number probem is $O(n^{1-\varepsilon})$-innaproximable in bipartite graphs in polynomial time unless P=NP.
- From MAX3SAT-INTERVAL

Geodetic convexity

- The results for hull number, Carathéodory number, interval number and convexity number can be extended to geodetic convexity

Geodetic convexity

Theorem

Let G_1 be a triangle free graph with at least three vertices. Then

(i) $hn_{gd}(G_1 + K_m) = hn_{P_3}(G_1)$,

(ii) $in_{gd}(G_1 + K_m) = in_{P_3}(G_1)$,

(iii) $cx_{gd}(G_1 + K_m) = cx_{P_3}(G_1) + m$,

(iv) $cth_{gd}(G_1 + K_m) = \max\{cth_{P_3}(G_1), 2\}$.

Geodetic convexity

- Geodetic Radon number probem is $O(n^{1-\varepsilon})$-innaproximable in general graphs in polynomial time unless P=NP.
- From MAXIMUM-CLIQUE problem

Thank you

- Thank you!