Property Testing and Parameter Testing for Permutations

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This is joint work with
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SODA 2010 (Austin-Texas, USA)
January 17, 2010 (9:25 - 9:45 AM) Session 1B

Basic definitions

A permutation σ on $[n] = \{1, 2, \ldots, n\}$ is a bijective function of the set $[n]$ into itself.

$(4, 5, 2, 3, 6, 1)$ is a permutation on $[6]$.

Objective

Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of permutations.

Question: When does $(\sigma_n)_{n\in\mathbb{N}}$ converge?

A natural property

The permutation τ on $[m]$ is a subpermutation of σ on $[n]$ if there is a subsequence of σ with same relative order of τ.

Example: $\tau = (3, 1, 4, 2)$, $\sigma = (5, 6, 2, 4, 7, 1, 3)$.

$$\sigma = ({\color{red}5}, 6, {\color{red}2}, 4, {\color{red}7}, 1, {\color{red}3}).$$

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Let $\Lambda(\tau, \sigma)$ be the number of occurrences of τ in σ. The density of the permutation τ as a subpermutation of σ is given by

$$t(\tau, \sigma) = \binom{n}{m}^{-1} \Lambda(\tau, \sigma).$$

Convergent permutation sequences

If τ is a fixed permutation and $(\sigma_n)_{n\in\mathbb{N}}$ is a convergent sequence, it would be natural to require that the real sequence $(t(\tau,\sigma_n))_{n\in\mathbb{N}}$ converges.

Definition

A sequence of permutations (σ_n) is said to converge (weakly) if, for every fixed permutation τ, the real sequence $(t(\tau,\sigma_n))_{n\in\mathbb{N}}$ converges.

Convergent permutation sequences

Example: Let σ_n be the identity permutation $= (1, 2, \ldots, n)$ on $[n]$.

$$t(\tau, \sigma_n) = \begin{cases} 1, & \text{if } \tau \text{ is an identity permutation of size } m \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

Example: Let π_n be a random permutation on $[n]$ (chosen uniformly).

$$\mathbb{E}(t(\tau, \pi_n)) = \begin{cases} 1/m!, & \text{if } |\tau| = m \leq n; \\ 0, & \text{if } m > n. \end{cases}$$

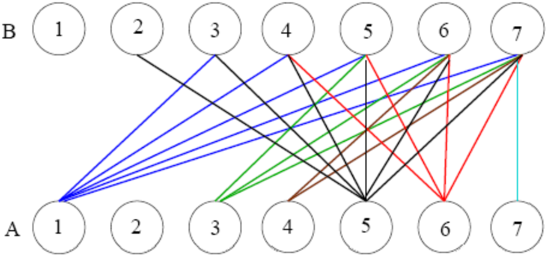
A limit for a permutation sequence?

Question: Is there a limit for a convergent permutation sequence?

Encoding permutations

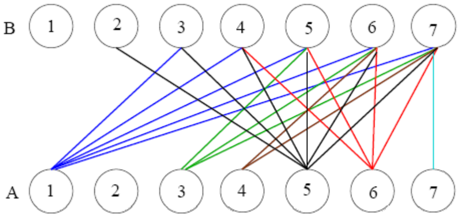
A permutation σ on $[n]$ can be encoded as a bipartite graph G_σ whose color classes A and B are disjoint copies of $[n]$, and where $(a, b) \in A \times B$ is an edge if and only if $\sigma(a) < b$.

$$\sigma = (2, 7, 4, 5, 1, 3, 6)$$



Weighted permutations

$\sigma = (2, 7, 4, 5, 1, 3, 6)$



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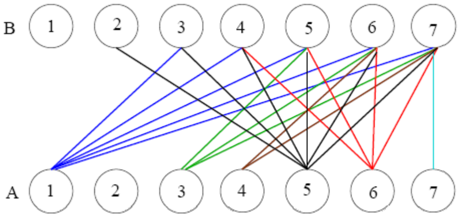

Weighted permutations

$\sigma = (2, 7, 4, 5, 1, 3, 6)$ and a partition $\mathcal{P}$ with three intervals.

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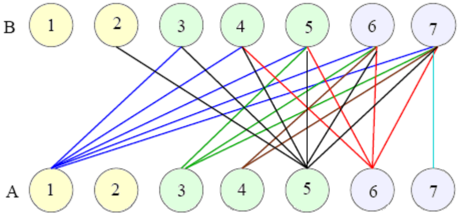

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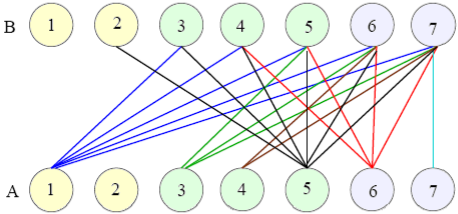

$$Q_{\sigma,\mathcal{P}} = \begin{bmatrix} 1/2 & 1 & 3/4 \\ 1/2 & 4/9 & 2/6 \\ 0 & 1/6 & 0 \end{bmatrix}$$

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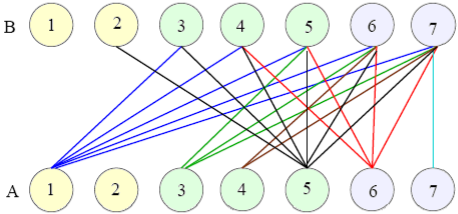

$$Q_{\sigma,\mathcal{P}} = \begin{bmatrix} 1/2 & 1 & 3/4 \\ 1/2 & 4/9 & 2/6 \\ 0 & 1/6 & 0 \end{bmatrix} \qquad \text{The lines sum} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \leq \begin{bmatrix} 9/4 \\ 23/18 \\ 1/6 \end{bmatrix} < \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

"Weighted permutation"

Limit permutations

Definition

A limit permutation is a Lebesgue measurable function $Z : [0,1]^2 \longrightarrow [0,1]$ satisfying:

(a) $Z(x,\cdot)$ is a cdf (cum.distr.funct.) continuous at 0 and 1 $(\forall x \in [0,1])$;

(b) $Z(\cdot,y)$ is a measurable function $(\forall y \in [0,1])$ s.t.

$$\int_0^1 Z(x,y)\,dx \;=\; y.$$

Density of subpermutations $\tau_{[m]}$ in $\sigma_{[n]}$

$$t(\tau,\sigma) = \binom{n}{m}^{-1} \sum_{x \in [n]^m_<} \sum_{y \in [n]^m_\tau} \prod_{i=1}^{m} \left(\sigma(x_i) = y_i\right),$$

$[n]^m_<$: $x = (x_1 < x_2 < \ldots < x_m)$ is increasing.
$[n]^m_\tau$: $y = (y_1, \ldots, y_m)$ has the same relative order of τ : $y_{\tau^{-1}(1)} < \ldots < y_{\tau^{-1}(m)}$.

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$$t(\tau,\sigma) = \binom{n}{m}^{-1} \sum_{x\in[n]^m_<} \sum_{y\in[n]^m_\tau} \prod_{i=1}^{m} \left(Q_\sigma(x_i, y_i + 1) - Q_\sigma(x_i, y_i)\right)$$

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$$t(\tau,\sigma) = m! \int_{x\in[0,1]^m_<} \left(\int_{y\in[0,1]^m_\tau} dZ_\sigma(x_1,\cdot)\cdots dZ_\sigma(x_m,\cdot)\right) dx_1\cdots dx_m$$

Density of subpermutations $\tau_{[m]}$ in Z

Definition

Given a limit permutation $Z : [0,1]^2 \to [0,1]$, the subpermutation density of τ in Z is given by

$$t(\tau, Z) \; = \; m! \int_{[0,1]^m_<} \Big(\int_{[0,1]^m_\tau} dZ(x_1, \cdot) \; \cdots \; dZ(x_m, \cdot) \Big) \; dx_1 \cdots dx_m$$

Existence of a limit

Theorem

Given a convergent sequence $(\sigma_n)_{n\in\mathbb{N}}$ of permutations, there exists a limit permutation $Z : [0,1]^2 \to [0,1]$ s.t.

$$\lim_{n\to\infty} t(\tau, \sigma_n) \;=\; t(\tau, Z)$$

for every permutation τ.

Uniqueness of the limit

Theorem

If Z_1 and Z_2 are limits to a sequence (σ_n), then they differ only in a set of measure zero.

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Theorem

Every limit permutation Z is the limit of a convergent sequence of permutations.

Random permutations

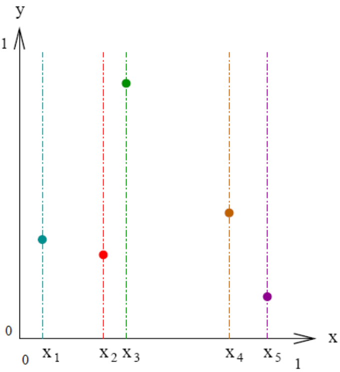
Given a limit permutation Z, the random permutation $\sigma(n, Z)$ is generated as follows.

- We generate a sequence $(x_1 < x_2 < \cdots < x_n)$ in $[0,1]^n_<$ uniformly

- We generate a sequence $(y_1, \ldots, y_n)$ in $[0,1]^n$, where y_k is generated according to the probability distribution $Z(x_k, \cdot)$

- $\sigma(n, Z)$ is given by the relative order of $(y_1, \ldots, y_n)$

Random permutations



(3,2,5,4,1)

Random permutations

Theorem

If Z is a limit permutation, then, with probability 1, the random sequence $(\sigma(n, Z))_{n\in\mathbb{N}}$ is convergent and its limit is Z.

Weak convergence × Strong convergence

A sequence of permutations $(\sigma_n)_{n \in \mathbb{Z}}$ is said to converge (strongly) if it is a Cauchy sequence with respect to the rectangular distance.

Theorem

$$\textit{strong convergence} \iff \textit{weak convergence}$$

Rectangular distance

Definition

Given permutations σ_1, σ_2 on $[n]$, the rectangular distance between σ_1 and σ_2 is given by

$$d_{\square}(\sigma_1, \sigma_2) = \frac{1}{n} \max_{S,T \in I[n]} \Big| |\sigma_1(S) \cap T| - |\sigma_2(S) \cap T| \Big|.$$

In particular, random permutations are close to each other with high probability.

Permutation parameters

Example: $fp(\sigma)$ is the number of fixed points of σ.

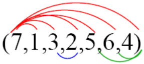
$$\sigma = (7, 1, 3, 2, 5, 6, 4) \qquad fp(\sigma) = 3$$

Example: $ord(\sigma)$ is the largest increasing subpermutation of σ.

$$\sigma = (7, 1, 3, 2, 5, 6, 4) \qquad ord(\sigma) = 4$$

Example: $inv(\sigma)$ is the number of inversions in σ.

$$\sigma = (7, 1, 3, 2, 5, 6, 4) \qquad inv(\sigma) = 9$$



Parameter testing

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Question: Can one accurately predict the value of a parameter $f(\sigma)$ in constant time for every permutation σ?

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Parameter Testing through subpermutations

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$sub(k,\sigma)$: random subpermutation of σ on $[k]$ (uniformly chosen)

$$\sigma = (5, 7, 2, 10, 1, 4, 8, 6, 3, 9) \qquad sub(4,\sigma) = (2, 4, 1, 3)$$

Parameter testing through subpermutations

Objective: accurately predict the value of a parameter $f(\sigma)$ by looking at a randomly chosen subpermutation of much smaller size.

Definition

A parameter f is testable if,
For every $\epsilon > 0$,
There exist positive integers $k < n_0$ s.t.:

If σ is a permutation of length $n > n_0$, then

$$\mathbb{P}\Big(|f(\sigma) - f(sub(k,\sigma))| > \varepsilon\Big) \;\leq\; \varepsilon.$$

Characterization of testable parameters

Theorem

A bounded permutation parameter is testable if and only if the sequence $(f(\sigma_n))_{n\in\mathbb{N}}$ converges for every convergent sequence $(\sigma_n)_{n\in\mathbb{N}}$ of permutations.

A permutation parameter f is bounded if there is a constant M such that $|f(\sigma)| < M$ for every permutation σ.

Immediate consequences

Testable Permutation Parameters

- The subpermutation density $f_\tau(\sigma) = t(\tau, \sigma)$ for any fixed τ.
- The inversion density $inv(\sigma) = t((2, 1), \sigma)$.

NOT Testable Permutation Parameters (through subpermutations)

- The fixed-point density.
- The density of a longest increasing subsequence.

Property testing through subpermutations

We now want to look at more general properties of a permutation:

- Does it satisfy a given condition?
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We now want to look at more general properties of a permutation:

- Does it satisfy a given condition?
- Does it contain or avoid a given set of patterns?

Question: Can one predict the answer of such a question accurately by looking at a small subpermutation?

Modified question: Can one at least predict accurately if a permutation σ satisfies a property $\mathcal{P}$ or is far from satisfying it by looking at a small subpermutation?

Property testing through subpermutations

More precisely: a permutation property $\mathcal{P}$ is testable if, for every $\epsilon > 0$, there exist $k \leq n_0$ s.t. if σ is a permutation on $[n]$ with $n \geq n_0$, then with probability $\geq 1 - \epsilon$:

(i) σ satisfies $\mathcal{P}$ $\implies$ $sub(k, \sigma)$ satisfies $\mathcal{P}$

(ii) σ is ϵ-far from satisfying $\mathcal{P}$ $\implies$ $sub(k, \sigma)$ does not satisfy $\mathcal{P}$

σ is ϵ-far from satisfying $\mathcal{P}$ if

$$d_{\square}(\sigma, \mathcal{P}) \quad = \min\{d_{\square}(\sigma, \pi) \ : \ \pi \text{ on } [n] \text{ satisfies } \mathcal{P}\} \ \geq \epsilon.$$

Hereditary properties

A permutation property $\mathcal{P}$ is hereditary if, whenever σ satisfies $\mathcal{P}$, then all its subpermutations satisfy $\mathcal{P}$.

Example: The property of avoiding a fixed pattern is hereditary.

Theorem

Every hereditary property is testable.

Conclusion

- We developed a theory for convergence of permutation sequences, along the lines of the theory introduced for graphs by Borgs, Chayes, Lovász, Sós, Szegedy and Vesztergombi.
- A limit object was identified. It is essentially unique and leads to a natural model of random permutations.
- This theory was applied to characterize a version of property testing and parameter testing in the permutation framework.

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