Spectral Graph Theory and Normal Modes of Coupled Oscillations

Rudini Sampaio (Universidade Federal do Ceará)

I. Guilhon, S. Madeiro (ITA, SJCampos, Brazil)
A.J.C. Sampaio, **R. Sampaio** (UFC, Fortaleza, Brazil)
Felipe Costa Cunha (IFCE, Maranguape, Brazil)

SBPO-2024
Fortaleza, Brazil
November 04-07

Spectral Graph Theory

It emerged in the 1960s and investigates “Linear Algebra” properties of matrices associated with the graph, such as the eigenvalues and the eigenvectors of the Adjacency Matrix A and the Laplacian Matrix L.

$$L = D - A$$

Why? Good question !! Hard intuition !!

The largest eigenvalue is closely related to average degree.

The second largest eigenvalue is closely related to connectivity

Spectral Graph Theory

It emerged in the 1960s and investigates “Linear Algebra” properties of matrices associated with the graph, such as the eigenvalues and the eigenvectors of the Adjacency Matrix A and the Laplacian Matrix L.

$$L = D - A$$

Why? Good question !! Hard intuition !!

Motivation from Electrical Networks [Kirchoff, 1847], Markov chains, etc…

Coupled Oscillations (Physics)

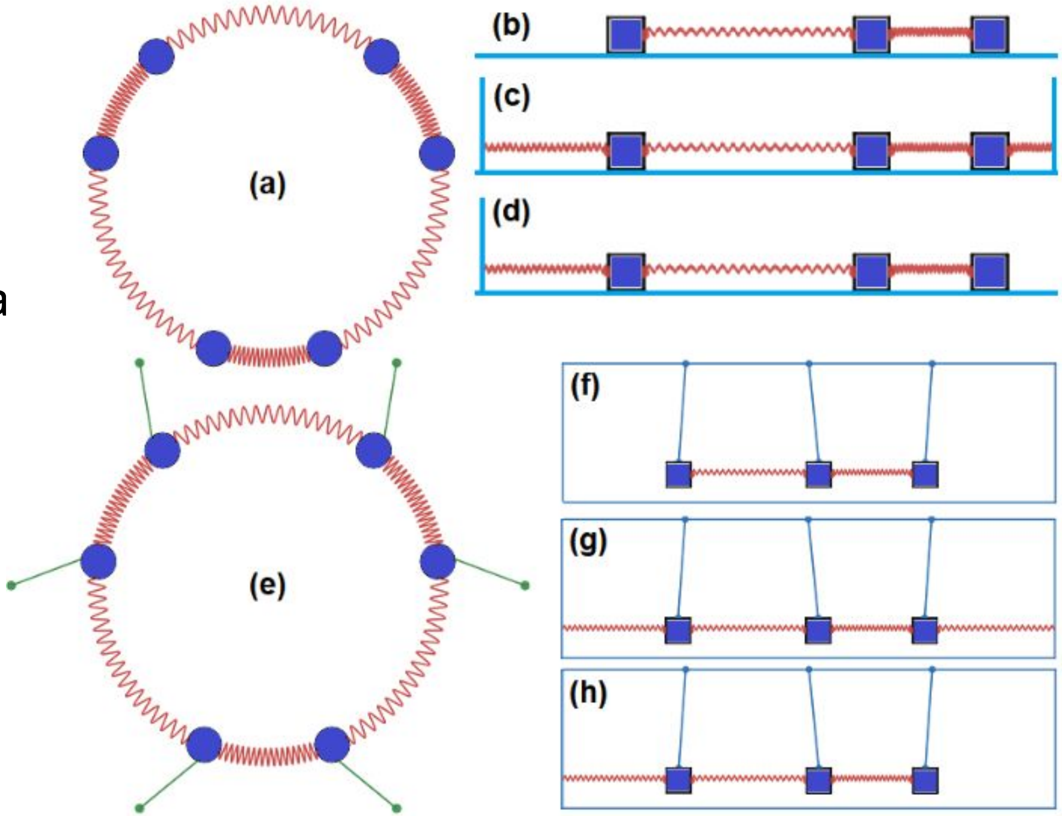
Interaction of oscillatory elements.

Normal modes: oscillations in which all elements have the same frequency.

Natural frequency: a frequency of a normal mode.

Most common example:
the spring-mass system

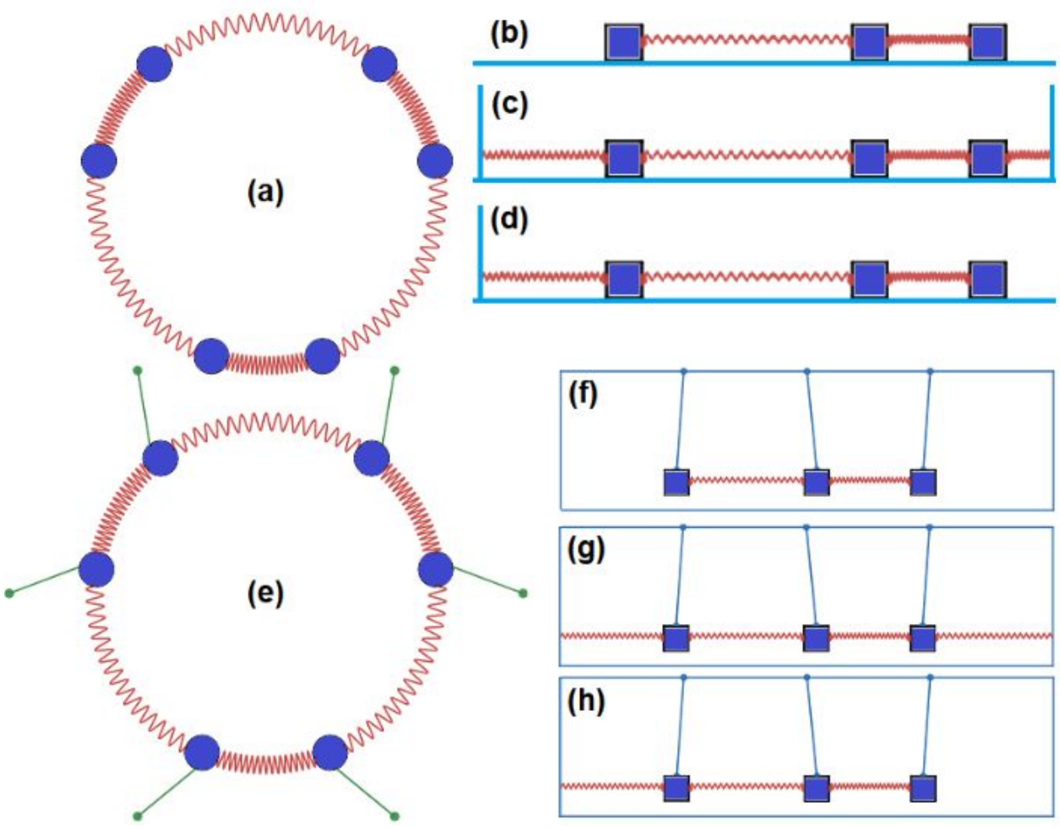
Every oscillation is a linear combination of the normal modes.



Coupled Oscillations (Physics)

Physics books generally presents such problems of coupled oscillations separately, providing extensive particular solutions, or not even mentioning the subject.

Few more advanced books present a Linear Algebra approach, involving eigenvalues and eigenvectors, but without a general model and just to a small number of blocks.



Diagonalization, Eigenvalues and Eigenvectors

A square matrix *Q* is diagonalizable if $Q = P \cdot D \cdot P^{-1}$ where D is a diagonal matrix

The matrix *D* contains the eigenvalues of *Q*. The matrix *P* contains eigenvectors of *Q*.

Equation of a Coupled Oscillation: $\ddot{X} = -Q \cdot X$, where *X* contains the *positions.*

$$P^{-1} \cdot \ddot{X} = -D \cdot P^{-1} \cdot X \Rightarrow \ddot{Z} = -D \cdot Z$$

where

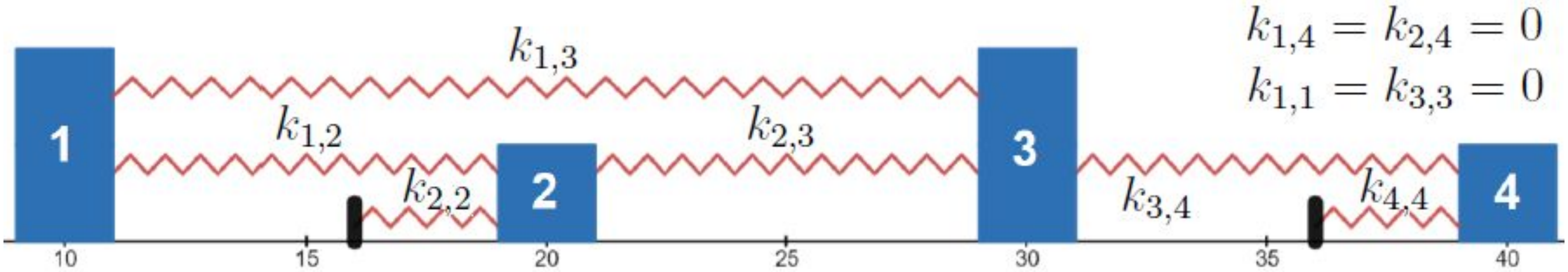
$$Z = P^{-1} \cdot X$$
$$X = P \cdot Z$$
$$x_i = \sum_{k=1}^{n} p_{i,k} \cdot z_k$$

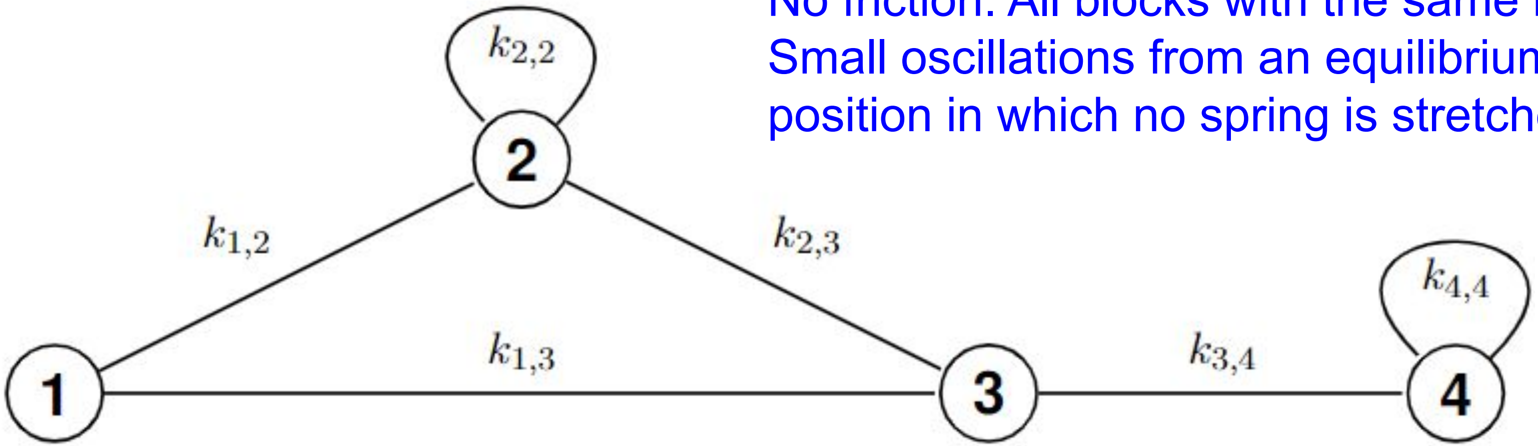
SHM (Simple Harmonic Motions): $\ddot{z}_i = -d_{i,i} \cdot z_i$

Angular frequency: $\omega_i = \sqrt{d_{i,i}}$ (sq. root of an eigenvalue). Frequency: divides by 2π

Eigenvectors are important to get the positions X given de normal mode Z.

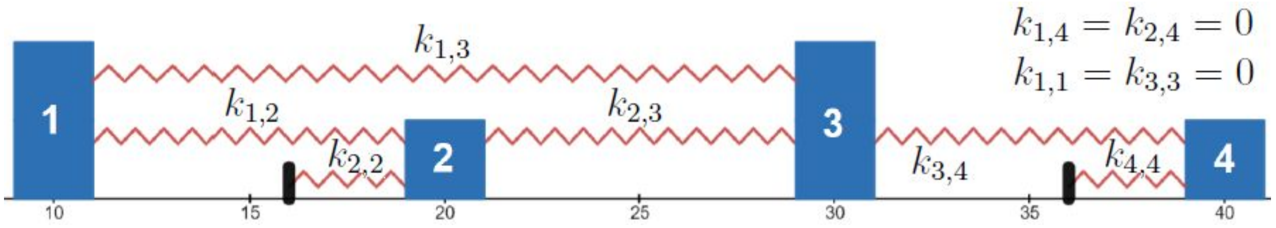
Contribution: General Model of Coupled Oscillation System





No friction. All blocks with the same mass *m*.
Small oscillations from an equilibrium position in which no spring is stretched.

Equations of the General Model of Coupled Oscillations



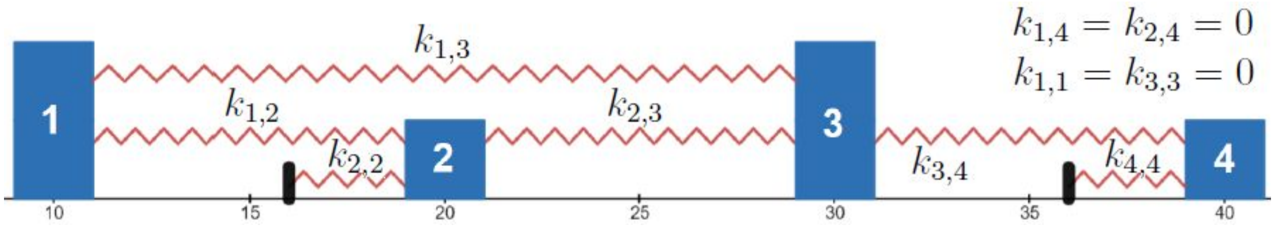
Let x_i be the deviation of the block *i* from its equilibrium position.

$$m \cdot \ddot{x}_i = -k_{i,i} \cdot x_i - \sum_{j=1}^{n} k_{i,j} \cdot (x_i - x_j) \quad \Rightarrow \quad m \cdot \ddot{x}_i = -\left(\sum_{j=1}^{n} k_{i,j}\right) \cdot x_i + \sum_{j \neq i}^{n} (k_{i,j} \cdot x_j)$$

$\ddot{x}_i = -\frac{1}{m} \cdot \sum_{j=1}^{n} L^*_{i,j} \cdot x_j$ Laplacian matrix L^* of the weighted graph

$\ddot{x}_i = -\frac{k}{m} \cdot \sum_{j=1}^{n} L_{i,j} \cdot x_j$ Laplacian matrix L of the "simple" graph

Equations of the General Model of Coupled Oscillations



Let x_i be the deviation of the block *i* from its equilibrium position.

Natural frequencies $\sqrt{\lambda_i^*/m}$ and $\sqrt{\lambda_i \cdot k/m}$ (eigenvalues of the Laplacian matrices).
Eigenvectors of the Laplacian matrices obtain the oscillations from the normal modes.

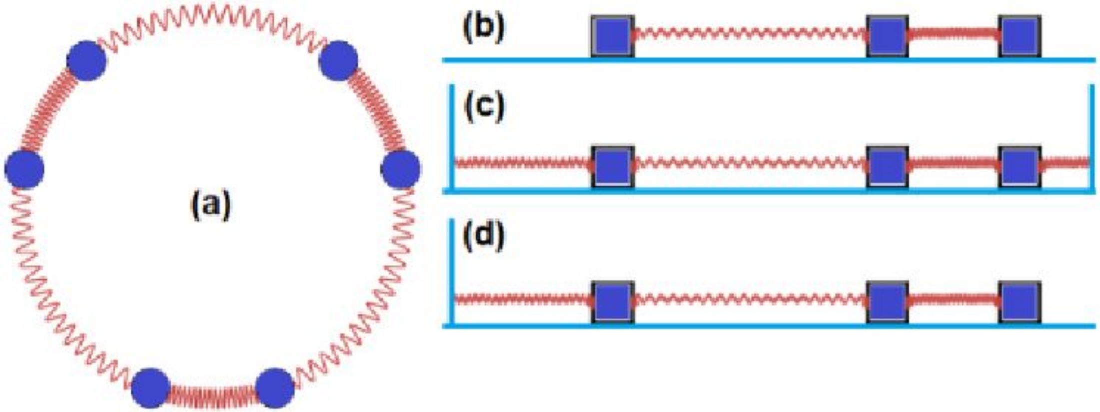
$$\ddot{X} = -\left(\frac{1}{m} \cdot L^*\right) \cdot X$$

Laplacian matrix L^* of the weighted graph

$$\ddot{X} = -\left(\frac{k}{m} \cdot L\right) \cdot X$$

Laplacian matrix L of the "simple" graph

Classical Systems (without pendulums)

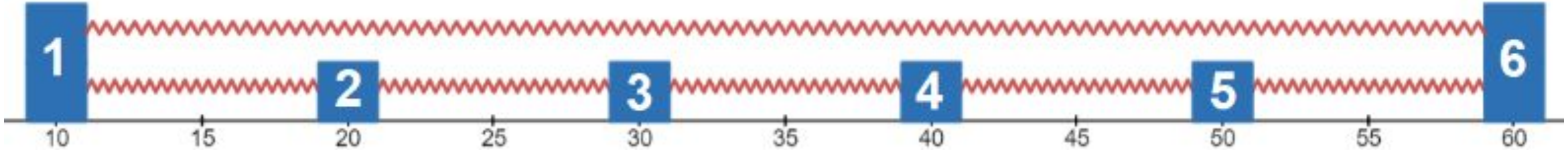


Spectral Graph Theory:
Laplacian of cycles and paths.

Linear system (b):

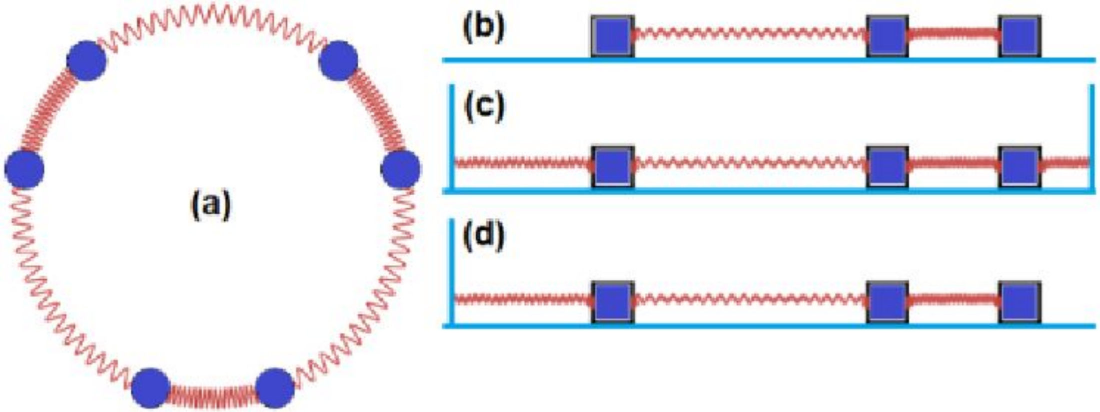
$$\omega_\ell = \sqrt{\frac{2k}{m}\cdot\left(1-\cos\left(\pi\cdot\frac{\ell}{n}\right)\right)}$$

$$for\ \ell = 1,\ldots,n-1$$



Circular system (a): $\omega_\ell = \sqrt{\frac{2k}{m}\cdot\left(1-\cos\left(2\pi\cdot\frac{\ell}{n}\right)\right)}$ for $\ell = 1,\ldots,\lfloor n/2\rfloor$

Classical Systems (without pendulums)



Spectral Graph Theory:
Laplacian of paths with 1 or 2 loops at the ends.

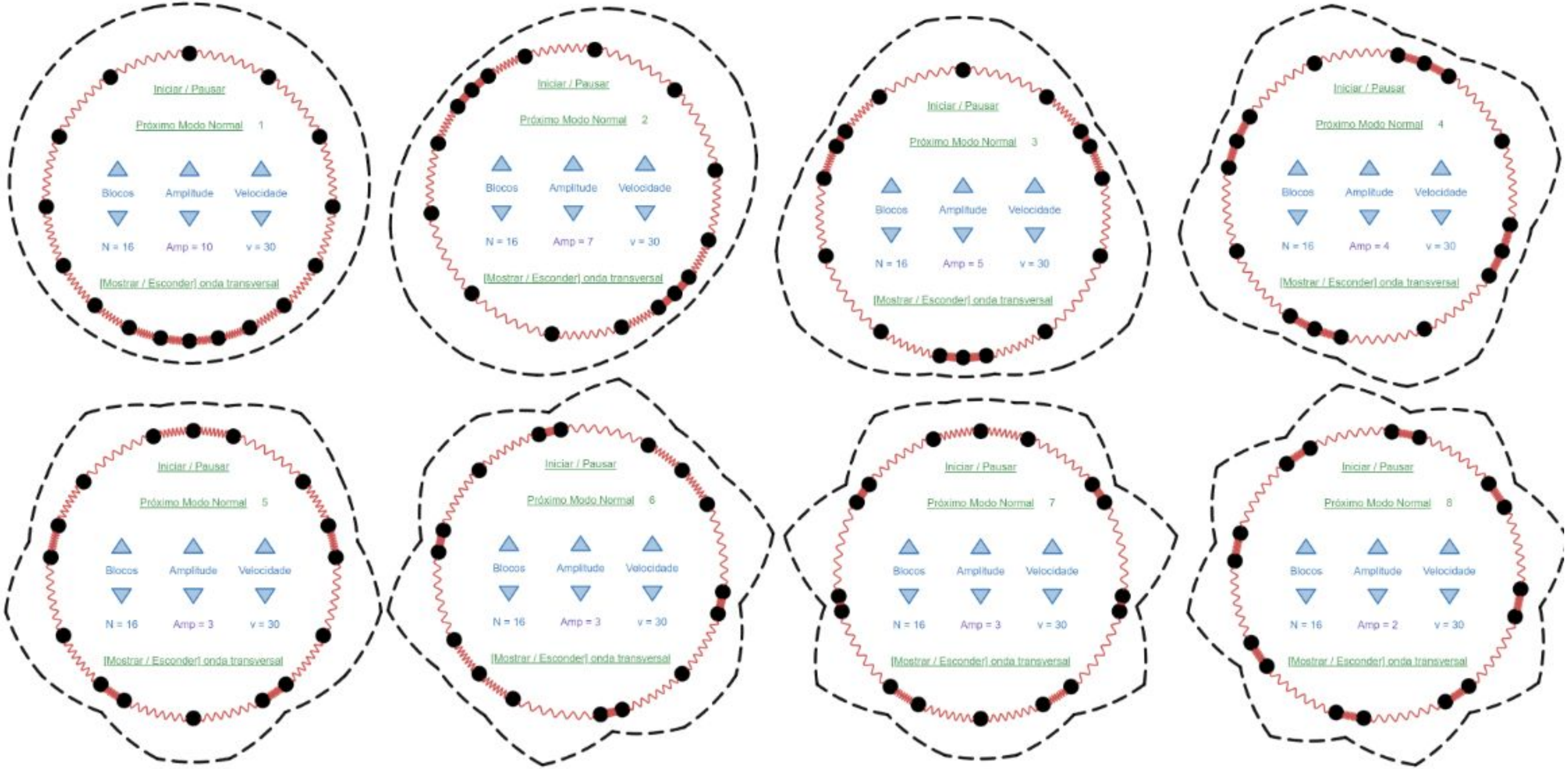
Linear with 2 walls (c):

$$\omega_\ell = \sqrt{\frac{2k}{m} \cdot \left(1 - \cos\left(\pi \cdot \frac{\ell}{n+1}\right)\right)} \quad \mathit{for}\ \ell = 1, \ldots, n$$

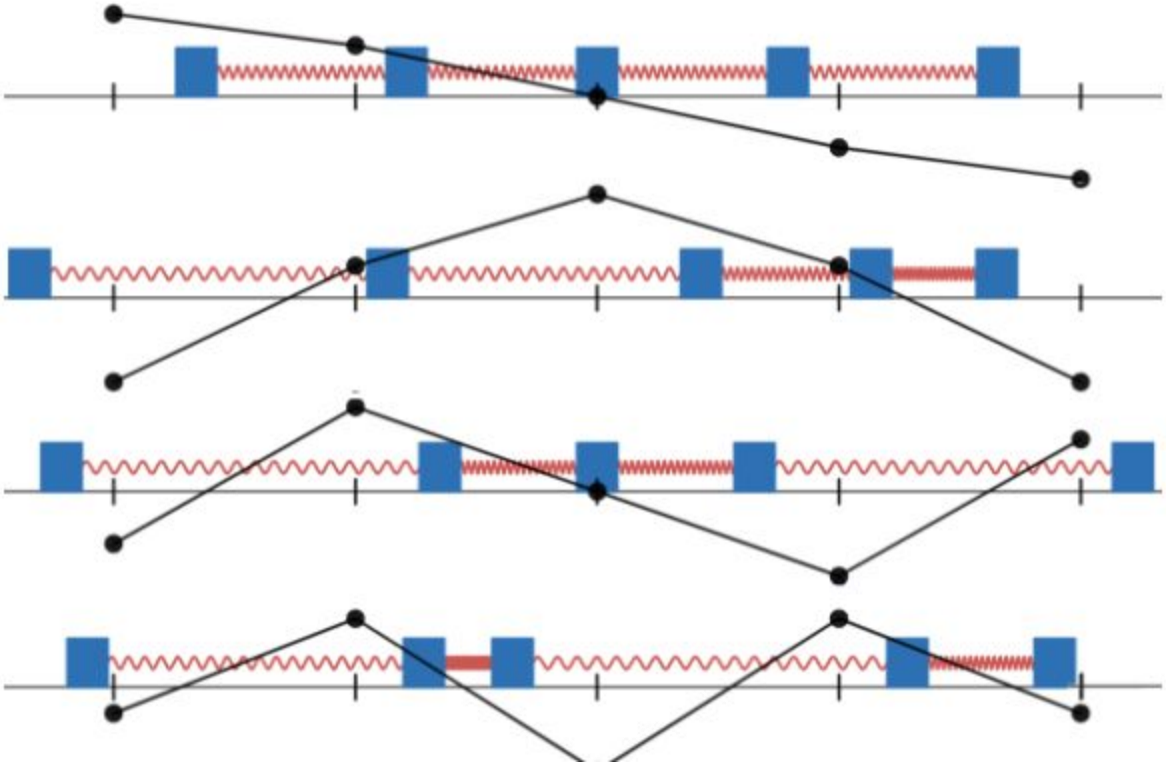
Linear with 1 wall (d):

$$\omega_\ell = \sqrt{\frac{2k}{m} \cdot \left(1 - \cos\left(\pi \cdot \frac{2\ell - 1}{2n+1}\right)\right)} \quad \mathit{for}\ \ell = 1, \ldots, n$$

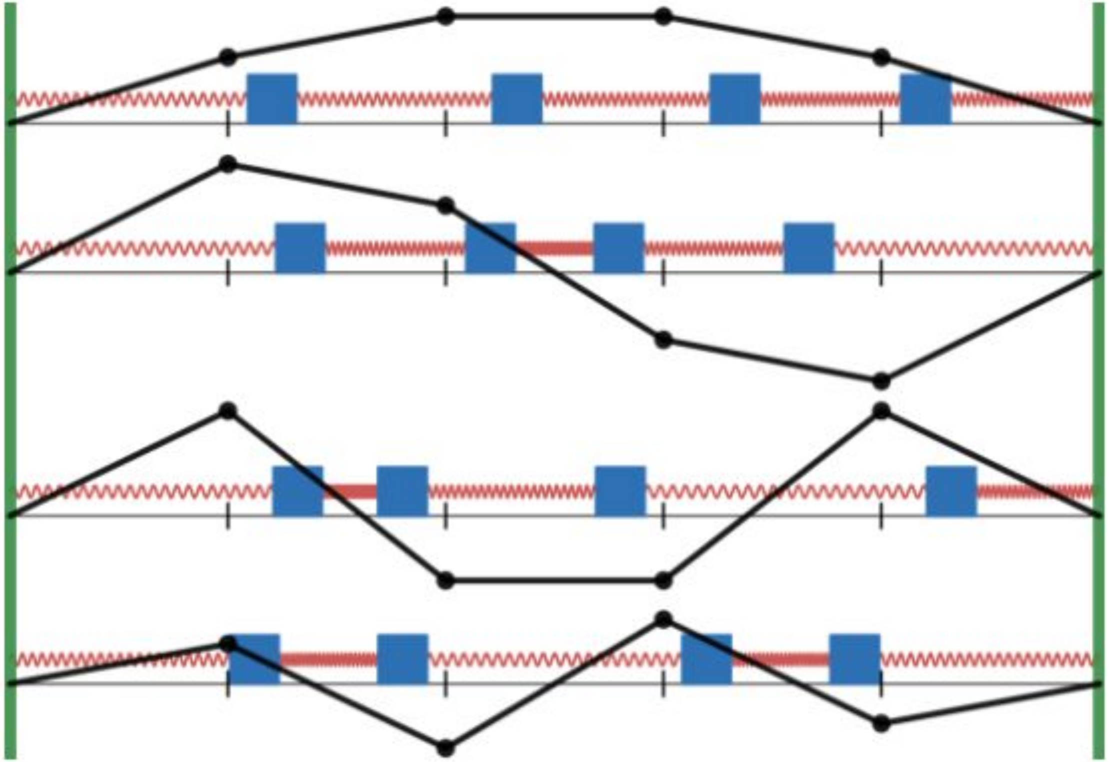
Visualization (Desmos graphing system): any number of blocks



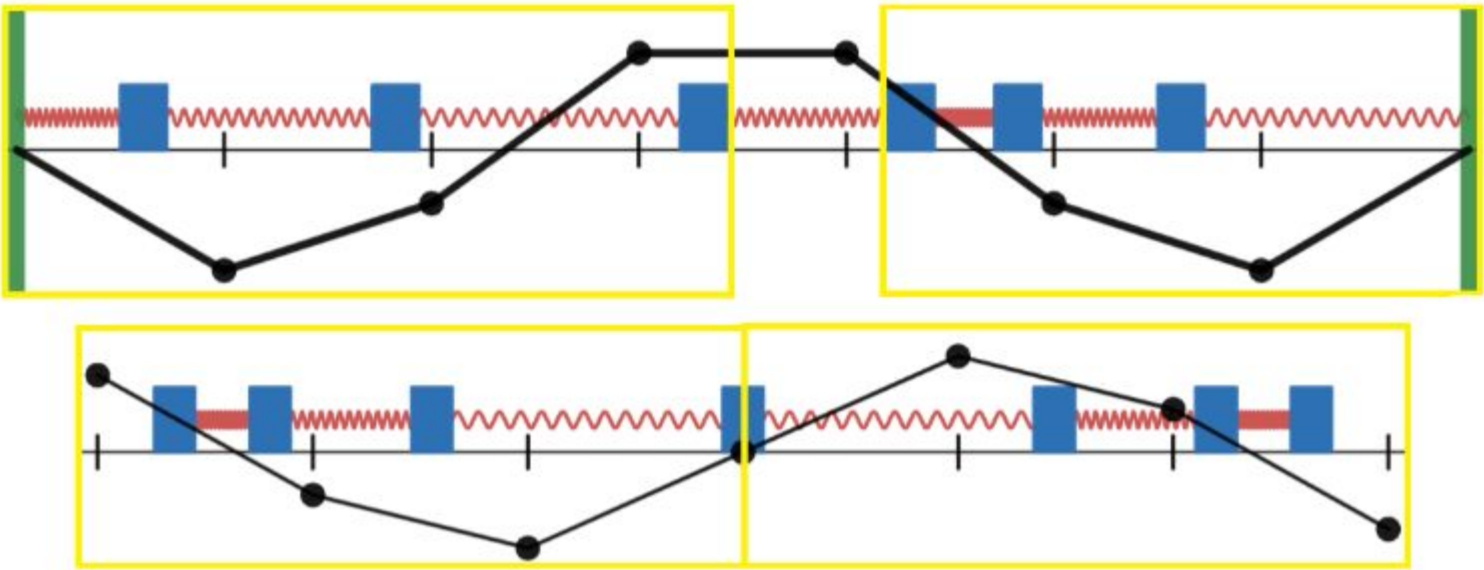
Visualization (Desmos graphing system): any number of blocks



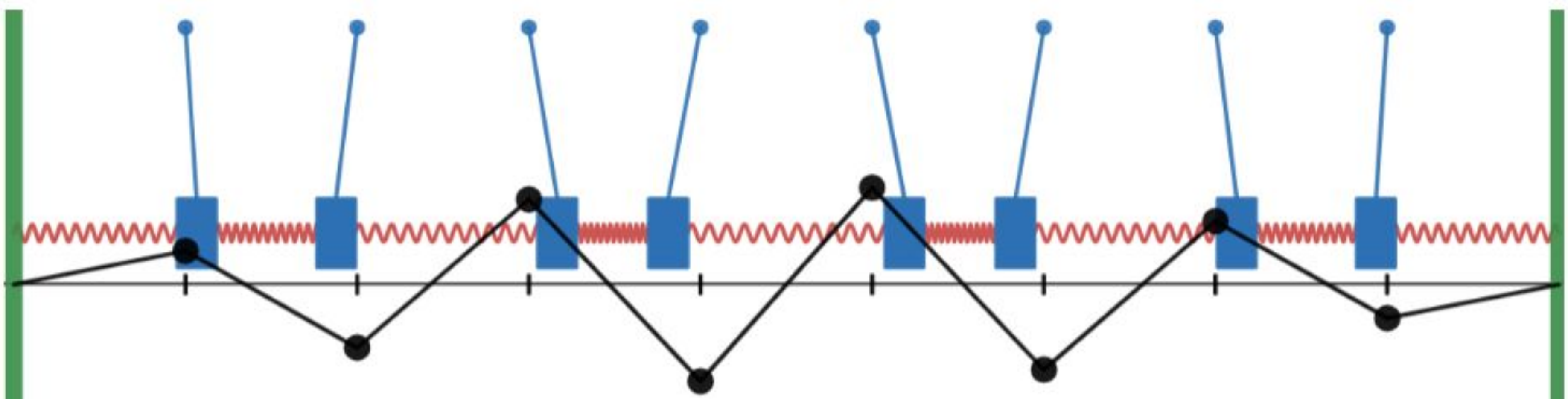
Visualization (Desmos graphing system): any number of blocks



Visualization (Desmos graphing system): any number of blocks



Classical Systems (with pendulums)



Small oscillations: restoring force $\approx m \cdot g \cdot \sin\theta_i = m \cdot g \cdot x_i/\ell$

Equivalent to a **spring in a wall** with constant $k' = m \cdot g/\ell$ for each vertex.

That is, every vertex of the graph has a **loop** with this weight.

Easy to calculate the eigenvalues and eigenvectors of the Laplacian matrix, given the ones of the graph without loops.

Consequences: Spectral Graph Theory ⇒ Physics

Eingenvalues of the Laplacian Matrix of the graph

⇕

Natural Frequencies of the Normal Modes of Oscillation

When the graph of Coupled Oscillation has Laplacian eigenvalues known by the Spectral Graph Theory, such as threshold graphs, we have the natural frequencies of oscillation.

Consequences: Spectral Graph Theory ⇐ Physics

Eingenvalues of the Laplacian Matrix of the graph

⇕

Natural Frequencies of the Normal Modes of Oscillation

Graph operations which maintains the natural frequencies of oscillation also maintain Laplacian eigenvalues

Example: Graph G obtained from N copies of a graph H where we can freely add edges between any pair of corresponding vertices in any two distinct copies of H.
Oscillation of H ⇒ Oscillation of G where the springs between copies does not stretch

Thanks, SBPO-2024