Jogos da Posição Convexa, do Intervalo e da Envoltória em Grafos

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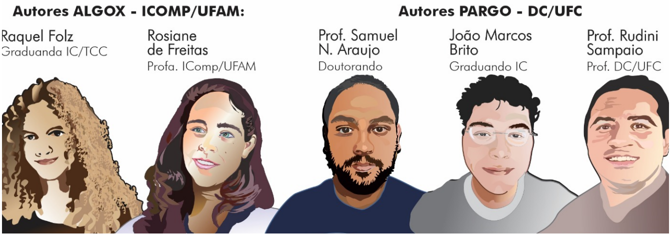
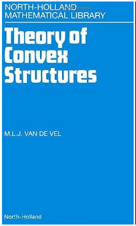
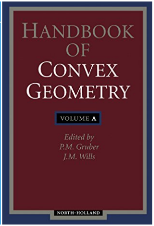
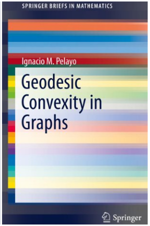
SBPO-2023, São José dos Campos, SP
Quarta, 08-Novembro, 18h

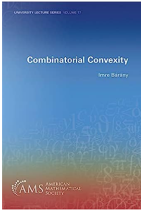


Convexidade em Grafos



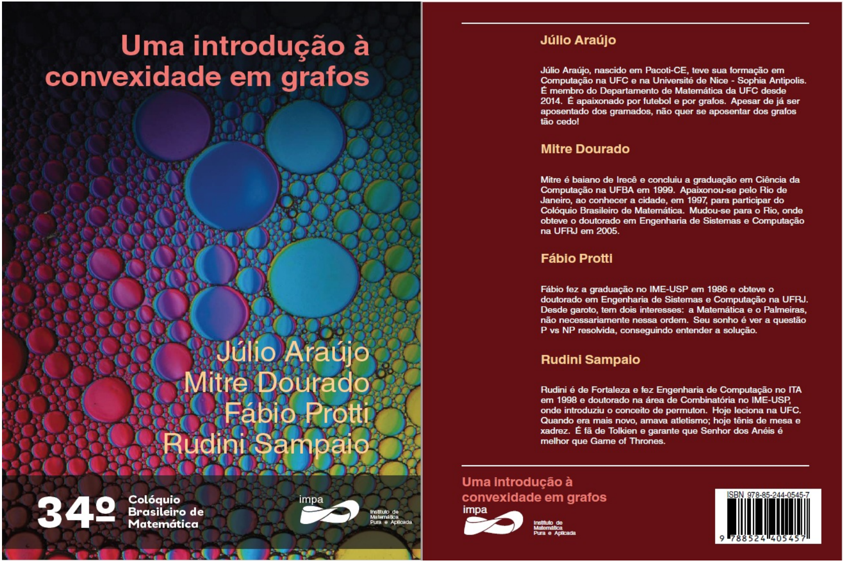






- Convexidade é um tema clássico da Matemática.
- Teoremas de Carathéodory (1911), Radon (1921), Helly (1923) e Erdős-Szekeres (1935): bases da área de Convexidade Combinatória.
- Harary-Nieminem'1981 *(Convexity in Graphs):* 1º artigo em grafos gerais, onde foi introduzido o parâmetro Tempo de Iteração.
- Harary'1984: Definição dos primeiros jogos de convexidade
- 2000-2010-2020: Pesquisa sobre a complexidade computacional de parâmetros de convexidade em grafos.
- Jayme Szwarcfiter: Grande pesquisador/propagador da área. Há 17 referências de artigos dele (citadas 30 vezes) em livro do IMPA, 2023.

Convexidade em Grafos



Os 10 Parâmetros de Convexidade de Grafos

1. hn(G): **número de Envoltória** tam menor conjunto de envoltória S de G na convexidade $\mathcal{C}$: $\mathrm{conv}_{\mathcal{C}}(S) = V$.
2. in(G): **número de Intervalo:** tam menor conjunto de intervalo S de G na convexidade $\mathcal{C}$: $\mathrm{I}_{\mathcal{C}}(S) = V$.
3. con(G): **núm Convexidade:** tam maior conjunto convexo $S \subsetneq V$
4. cth(G): **número de Carathéodory**
5. rd(G): **número de Radon**
6. hℓ(G): **número de Helly**
7. rk(G): **Posto (rank)**
8. gp(G): **número de Posição Geral**
9. ti(G): **Tempo de Iteração:** max $\mathrm{ti}_{\mathcal{C}}(S)$ entre conjuntos $S \subseteq V$.
10. tp(G): **Tempo de Percolação:** max $\mathrm{ti}_{\mathcal{C}}(S)$ entre conjuntos $S \subseteq V$, que são conjuntos de envoltória. $\mathrm{ti}(G) \leq \mathrm{tp}(G)$

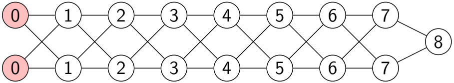
$$\mathrm{gp}(G) \geq \mathrm{rk}(G) \geq \mathrm{rd}(G) \geq \mathrm{h}\ell(G) \geq \frac{\mathrm{rd}(G)}{\mathrm{cth}(G)}$$

Convexidade em Grafos

Convexidade Geodésica

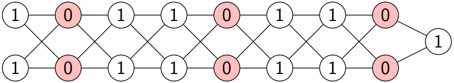
- Seja G um grafo e $S \subseteq V(G)$. O *intervalo geodésico* $\mathrm{I_g}(S)$ é o conjunto S e todo vértice em caminho mínimo entre 2 vértices de S.
- S é *convexo geodesicamente* se $\mathrm{I_g}(S) = S$. O *fecho convexo geodésico* $\mathrm{conv_g}(S)$ é o menor conjunto convexo contendo S.
- Aplicações sucessivas de $\mathrm{I_g}(\cdot)$ sobre $S \quad \Rightarrow \quad \mathrm{conv_g}(S)$

G



- $S \subseteq V(G)$ é conjunto de intervalo se $\mathrm{I_g}(S) = V(G)$
- $S \subseteq V(G)$ é conjunto de envoltória se $\mathrm{conv_g}(S) = V(G)$

H

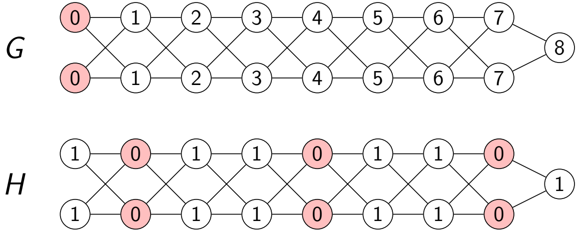


- **Convexidade Monofônica:** caminhos induzidos
- **Convexidade P_3:** caminhos com 3 vértices. P_3^*: caminhos induzidos

Convexidade em Grafos

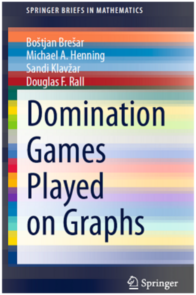
Convexidades Geodésica, Monofônica e P_3/P_3^*

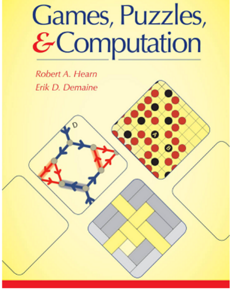
- **Geodésica:** caminhos mínimos. $\mathrm{I_g}(S)$, $\mathrm{conv_g}(S)$
- **Monofônica:** caminhos induzidos. $\mathrm{I_m}(S)$, $\mathrm{conv_m}(S)$
- P_3**:** caminhos com 3 vértices. $\mathrm{I_{p3}}(S)$, $\mathrm{conv_{p3}}(S)$
- P_3^***:** caminhos induzidos com 3 vértices. $\mathrm{I_{p3*}}(S)$, $\mathrm{conv_{p3*}}(S)$

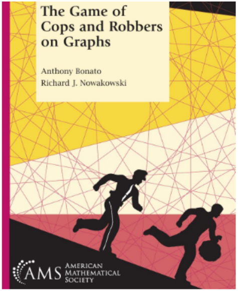


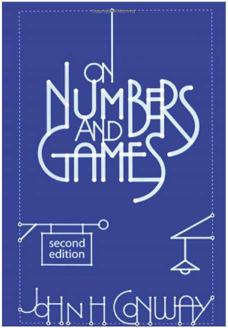
- $S \subseteq V(G)$ é conjunto de intervalo se $\mathrm{I_g}(S) = V(G)$
- $S \subseteq V(G)$ é conjunto de envoltória se $\mathrm{conv_g}(S) = V(G)$

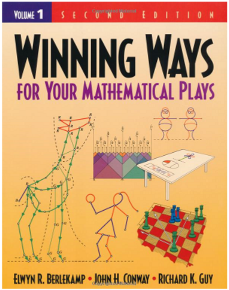
Combinatorial games

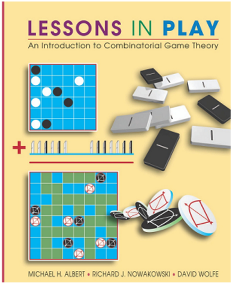












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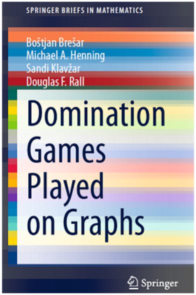

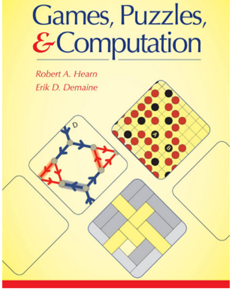

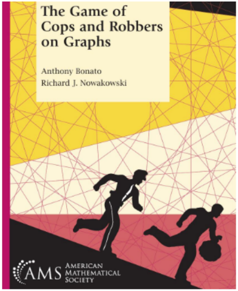

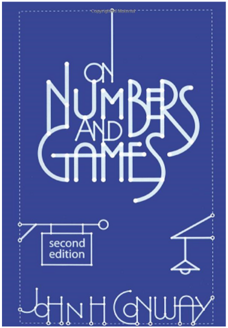

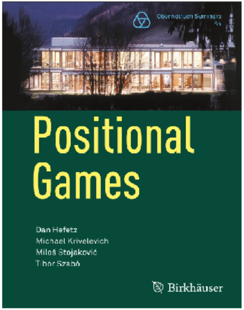

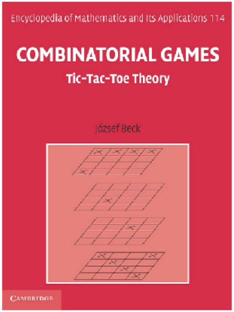

What is a Combinatorial Game?

- Two-person games with perfect information and no chance moves.
- Initial conditions: initial position + the first player.
- Players alternate moves.   Outcome: win, lose or tie (draw).
- Set of Terminal positions: from which no moves are possible.
- Finite games: rules to avoid loops (repetition of a position).

Variants of a Combinatorial Game

- **Normal variant:** the last to play wins (**achievement**).
- **Misère variant:** the last to play loses (**avoidance**).

Main question (decision problem)

- Zermelo-von Neumann Thm (1913): Given an instance of a game without draw, one player has a winning strategy.
- Does the 1st (first) player have a winning strategy?

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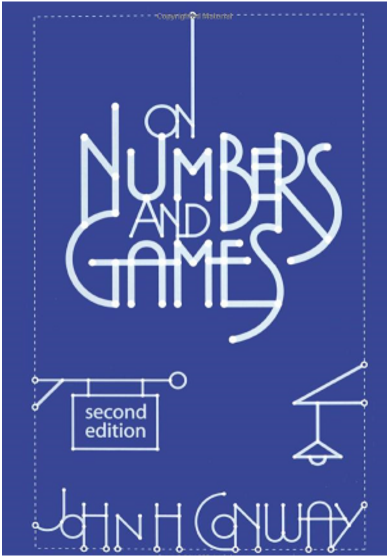
Classification of Combinatorial Games

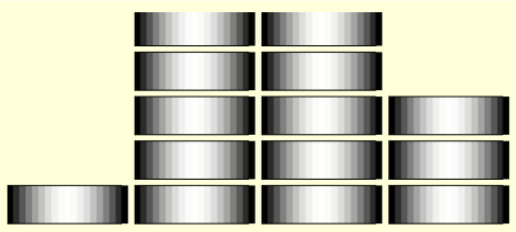
- **Impartial:** both players have the same set of possible moves and same objectives at any turn. The only difference is the first to play.
- **Partizan:** non-impartial games.

Impartial Games can be represented by numbers



Impartial Games

- **Sprague-Grundy Theory (1936):** Every finite impartial game in normal play is equivalent to one-heap game of **Nim**, and then can be represented by a number. **Nim** is polynomial time solvable.
- Nim game (Bouton, 1901): n heaps with $k_1, \ldots, k_n$ objects. The player chooses a heap and remove any number of objects from it.
- The solution uses the bitwise-xor operation.



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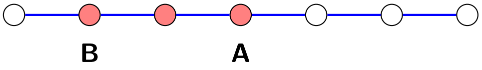
Nim/Geodesic



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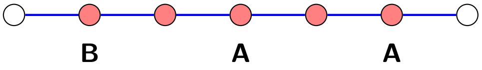

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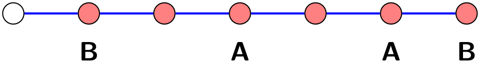

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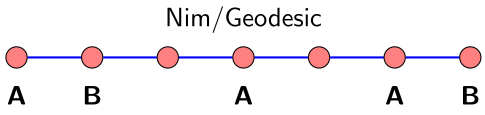

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Convexity Games

Annals of Discrete Mathematics 20 (1984) 323
North-Holland

CONVEXITY IN GRAPHS: ACHIEVEMENT AND AVOIDANCE GAMES

Frank HARARY

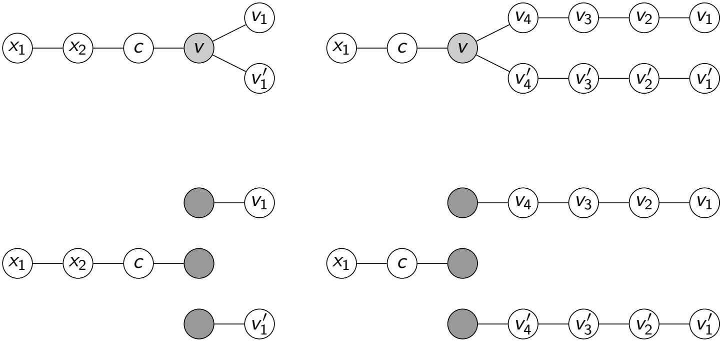
University of Michigan, Ann Arbor, Michigan, U.S.A.

Reference

[1] F. Harary and J. Nieminen, Convexity in graphs, J. Differential Geometry 16 (1981) 185–190.

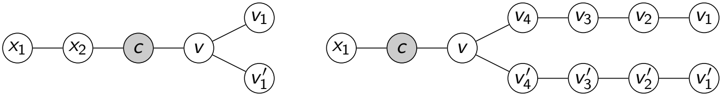
Convexity games: hull game and interval game

- Jogadores selecionam vértices até formar um conjunto de envoltória (ou de intervalo).
- No jogo geral, só se proíbe selecionar vértices já selecionados.
- No jogo fechado, é proibido selecionar vértices na envoltória (ou no intervalo) dos vértices já selecionados.
- Variante normal e misère: Alice perde o jogo fechado se jogar em v



closed Hull game in trees

- Teoria de Sprague-Grundy para jogos imparciais
- Cada configuração é associada a um número (**nimber**) ≥ 0
- União disjunta: operação *bitwise-xor* (ou-exclusivo bit a bit).
- Operação **mex**: menor valor não-negativo não presente entre os valores das configurações obtidas após uma jogada.



- Variante normal e misère: Alice vence o jogo fechado se jogar em c
- $1 \oplus 1 = 0; \quad 1 \oplus 0 = 1 \quad \Rightarrow \quad mex(\{0,1\}) = 2$
- $4 \oplus 4 = 0; \quad 4 \oplus 3 = 7; \quad 4 \oplus 2 = 6; \quad 4 \oplus 1 = 5; \quad 4 \oplus 0 = 4 \quad \Rightarrow$ $\mathbf{mex}(\{0,4,5,6,7\}) = 1$

- jogo simplificado: além do grafo, é dado um vértice que já deve estar selecionado antes do início do jogo.

closed Hull game in trees - polynomial algorithm

Algorithm 1 SIMPLIFIED-CIG$_{\mathrm{g}}$-leaf(T, v, u)

if $nimb[u]$ was already calculated **then return** $nimb[u]$
if u has degree 1 in T **then** $nimb[u] \leftarrow 1$; **return** 1
let $u_1, \ldots, u_k$ be the neighbors of u distinct from v in T
for $i = 1, \ldots, k$ **do**
    SIMPLIFIED-CIG$_{\mathrm{g}}$-leaf(T, u, u_i)
    let $h_i \leftarrow nimb[u_i]$
if $k > 1$ **then**
    $N \leftarrow \{h_1 \oplus \ldots \oplus h_k\}$
    for $i = 1, \ldots, k$ **do**
        for $h_i' = 0, \ldots, h_i - 1$ **do**
            $N \leftarrow N \cup \{h_1 \oplus \ldots \oplus h_i' \oplus \ldots \oplus h_k\}$
    $nimb[u] \leftarrow \mathrm{mex}(N)$; **return** $nimb[u]$
else
    $nimb[u] \leftarrow h_1 + 1$; **return** $nimb[u]$

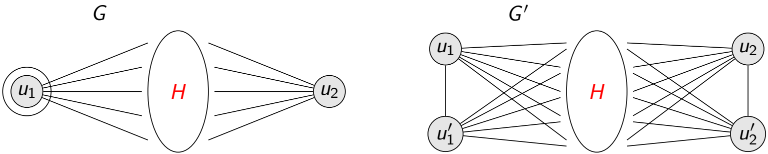
closed Hull game in trees - polynomial algorithm

Algorithm 2 SIMPLIFIED-CIG$_g$-tree(T, v)

let $h \leftarrow 0$ and **let** $v_1, \ldots, v_k$ be the neighbors of v in T
for $i = 1, \ldots, k$ **do**
    $h \leftarrow h \oplus$ SIMPLIFIED-CIG$_g$-leaf(T, v, v_i)
if (normal play) **then**
    if $h > 0$ **then** Alice wins **else** Bob wins
if (misère play) **then**
    if all subtrees of v has exactly one non-labeled vertex **then**
        if $h = 0$ **then** Alice wins **else** Bob wins
    else
        if $h > 0$ **then** Alice wins **else** Bob wins

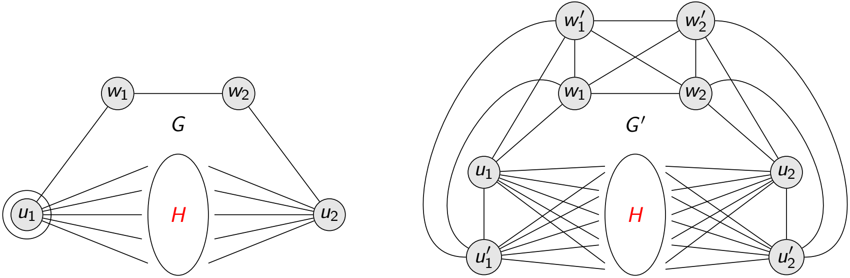
misère Hull game is PSPACE-complete

- Provamos que Hull game (geral ou fechado ou simplificado) é PSPACE-completo.
- Redução do jogo de formação de cliques: Os jogadores selecionam vértices de H que devem formar uma clique com os vértices já selecionados.
- Esse jogo é PSPACE-completo [Schaefer, 1978]



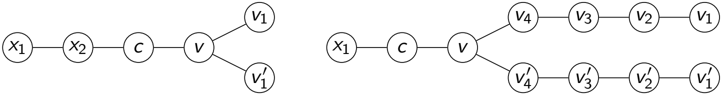
normal Hull game is PSPACE-complete

- Provamos que Hull game (geral ou fechado ou simplificado) é PSPACE-completo.
- Redução do jogo de formação de cliques: Os jogadores selecionam vértices de H que devem formar uma clique com os vértices já selecionados.
- Esse jogo é PSPACE-completo [Schaefer, 1978]



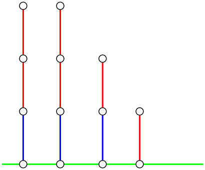
Hull game in Ptolemaic graphs

- Jogo não-fechado: única proibição são vértices já selecionados

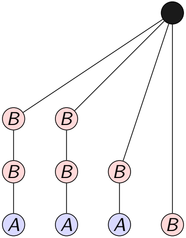


- $v \in S \subset V(G)$ é *extremo* de S se $S \setminus \{v\}$ também é convexo.
- [Farber-Jamison,1986]: v é extremo na convexidade geodésica se e só se é simplicial (sua vizinhança fechada é uma clique).
- [Farber-Jamison,1986]: uma convexidade é *geométrica* (*geometria convexa*) se satisfaz a propriedade de *Minkowski-Krein-Milman*: todo conjunto convexo é o fecho convexo de seus pontos extremos.
- A pesquisa em convexidades geométricas se concentra em determinar a classe de grafos para a qual certa convexidade é geométrica. Monofônica (cordal), Geodésica (Ptolemaico) e P_3 (floresta estrelas)
- **Teorema:** Alice vence o Hull Game em um grafo Ptolemaico se e só se n é ímpar (variante normal) ou n é par (misère).
- Melhoria de um artigo de 2003 para grafos bloco.

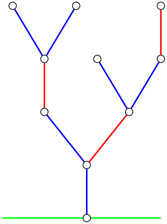
Jogos partizan de convexidade

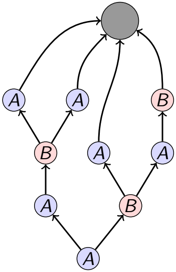


$\frac{1}{4} + \frac{1}{4} \ + \ \frac{1}{2} - 1 \ = \ 0$

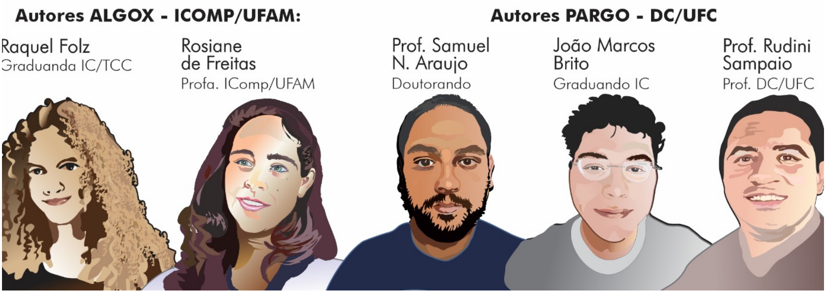


- Conway's Combinatorial Game Theory
- Clássico jogo Blue-Red Hackenbush





Dúvidas ???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