

Identifying codes in hexagonal grids

Rudini Sampaio

Universidade Federal do Ceará (UFC)
Fortaleza, Brazil

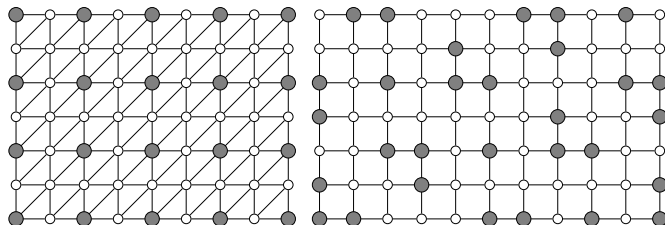
This is a joint work with

Gabriel Sobral (IME-USP, São Paulo, Brazil)
Yoshiko Wakabayashi (IME-USP, São Paulo, Brazil)

PARGO, 28-Abril-2023, 13h-14h

Identifying code

- ▶ Grafo G e um conjunto $C \subseteq V(G)$
- ▶ Vizinhança fechada de um vértice v : $N[v] = N(v) \cup \{v\}$.
- ▶ C -código de v : $C[v] = N[v] \cap C$
- ▶ C é um código de identificação se todos os vértices tem C -códigos não-vazios distintos.

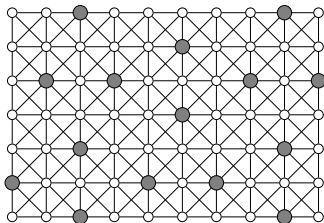


(a) Triangular grid \mathcal{G}_T : density $1/4$

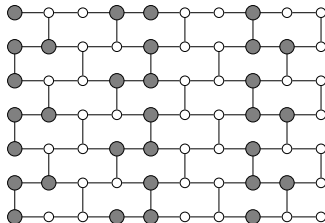
(b) Square grid \mathcal{G}_S : density $7/20$

Identifying code

- ▶ Grafo G e um conjunto $C \subseteq V(G)$
- ▶ Vizinhança fechada de um vértice v : $N[v] = N(v) \cup \{v\}$.
- ▶ C -código de v : $C[v] = N[v] \cap C$
- ▶ C é um código de identificação se todos os vértices tem C -códigos não-vazios distintos.



(c) King grid \mathcal{G}_K : density $2/9$



(d) Hexagonal grid \mathcal{G}_H : density $3/7$

Identifying code

- ▶ Grafo G e um conjunto $C \subseteq V(G)$
- ▶ Vizinhança fechada de um vértice v : $N[v] = N(v) \cup \{v\}$.
- ▶ C -código de v : $C[v] = N[v] \cap C$

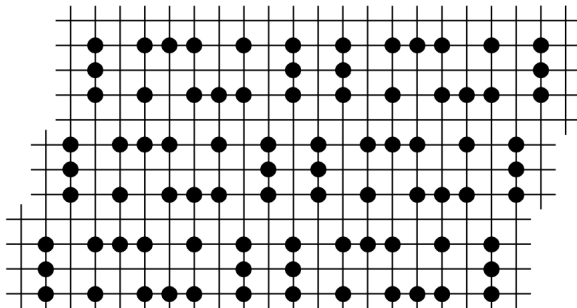


Figura: Código do grid retangular com densidade $7/20$

Identifying code

- ▶ Grafo G e um conjunto $C \subseteq V(G)$
- ▶ Vizinhança fechada de um vértice v : $N[v] = N(v) \cup \{v\}$.
- ▶ C -código de v : $C[v] = N[v] \cap C$
- ▶ C é um código de identificação se todos os vértices tem C -códigos não-vazios distintos.

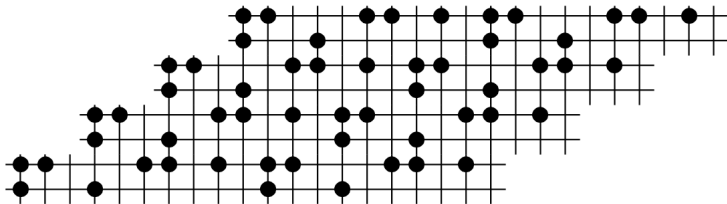


Figura: Outro código do grid retangular com densidade $7/20$

Known Results

Grid triangular \mathcal{G}_T :

- ▶ [Karpovsky et al., 1998] $d(\mathcal{G}_T) = 1/4$.

Grid rectangular \mathcal{G}_S :

- ▶ [Cohen et al., 1999] $d(\mathcal{G}_S) \leq 7/20$.
- ▶ [Ben-Haim, Litsyn, 2005] $d(\mathcal{G}_S) = 7/20$.
- ▶ [Daniel, Gravier, Moncel, 2004] $d(S_1) = 1/2$, $d(S_2) = 3/7$,
 $\frac{7}{20} - \frac{1}{2k} \leq d(S_k) \leq \min \left\{ \frac{2}{5}, \frac{7}{20} + \frac{2}{k} \right\}$
- ▶ [Bouznif et al., 2014] $d(S_3) = 3/7$,
 $\frac{7}{20} + \frac{1}{20k} \leq d(S_k) \leq \min \left\{ \frac{2}{5}, \frac{7}{20} + \frac{3}{10k} \right\}$.

King Grid \mathcal{G}_K :

- ▶ [Charon et al., 2002] $d(\mathcal{G}_K) = 2/9$.

Grid hexagonal \mathcal{G}_H :

- ▶ [Cohen et al., 2000] $16/39 \leq d(\mathcal{G}_H) \leq 3/7$.
- ▶ [Cranston, Yu, 2009] $12/29 \leq d(\mathcal{G}_H) \leq 3/7$.
- ▶ [Cukierman, Yu, 2013] $5/12 \leq d(\mathcal{G}_H) \leq 3/7$.

Results on Triangular Grids with k rows

[Rennan, Havet, Rudini, 2017]

- ▶ Triangular grid T_k with k rows:
- ▶ $d(T_2) = 1/2$
- ▶ $d(T_3) = 1/3$
- ▶ $d(T_4) = 1/3$
- ▶ $d(T_5) = 3/10$
- ▶ $d(T_6) = 1/3$
- ▶ $d(T_k) = 1/4 + 1/(4k)$ for all $k \geq 7$ odd
- ▶ $1/4 + 1/(4k) \leq d(T_k) \leq 1/4 + 1/(2k)$ for all $k \geq 8$ even.

Triangular grid with 2 rows

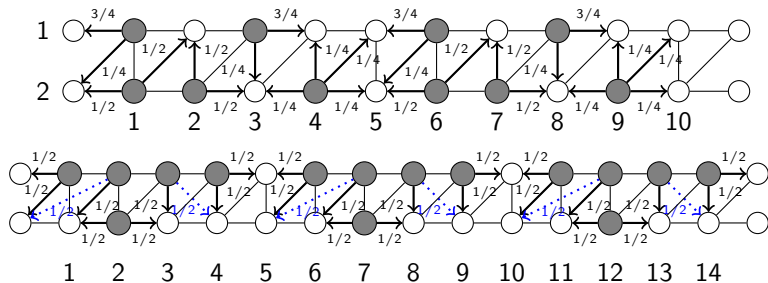


Figura: Identifying codes of T_2 with density $1/2$

Triangular grid with 3 rows

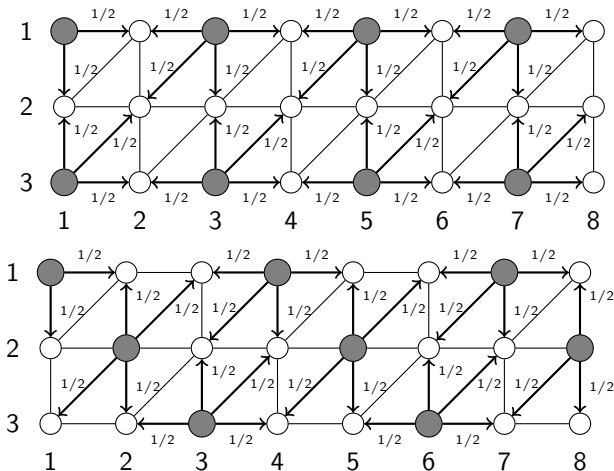


Figura: Identifying codes of T_3 with density $1/3$

Triangular grid with 4 rows

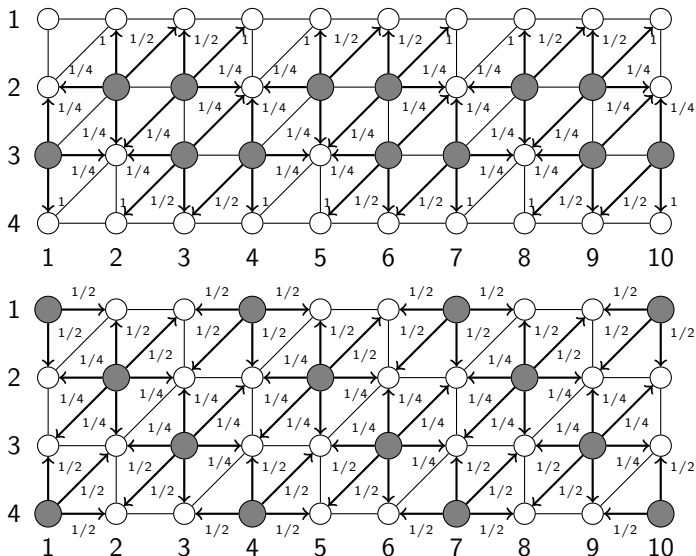


Figura: Two identifying codes of T_4 with density $1/3$

Triangular grid with 5 rows or more (odd)

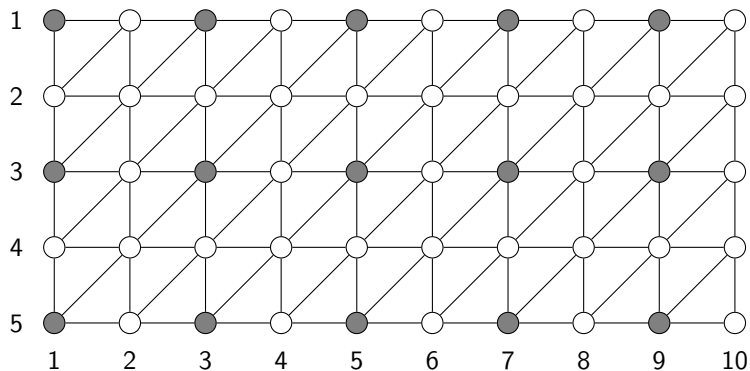


Figura: Identifying code C_5 of T_5 with density $3/10$.

Triangular grid with 6 rows or more (even)

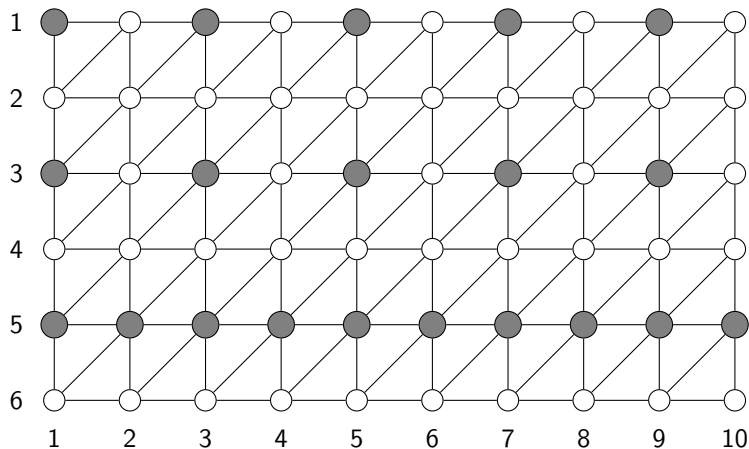


Figura: Identifying code \mathcal{C}_6 of T_6 with density $1/3$

Results on King grids with k rows

[Rennan, Havet, Rudini, 2018]:

- ▶ King Grid R_k with k rows
- ▶ $d(R_3) = 1/3 = 0,3333\dots$
- ▶ $d(R_4) = 5/16 = 0,3125$
- ▶ $d(R_5) = 4/15 = 0,2666\dots$
- ▶ $d(R_6) = 5/18 = 0,2777\dots$
- ▶ $d(R_n) \geq 2/9 + 5/81n$, for every $n \geq 7$
- ▶ $d(R_n) \leq 2/9 + 6/18n$, for every $n = 3k \geq 9$
- ▶ $d(R_n) \leq 2/9 + 8/18n$, for every $n = 3k + 1 \geq 7$
- ▶ $d(R_n) \leq 2/9 + 7/18n$, for every $n = 3k + 2 \geq 8$

King grid with 3 rows

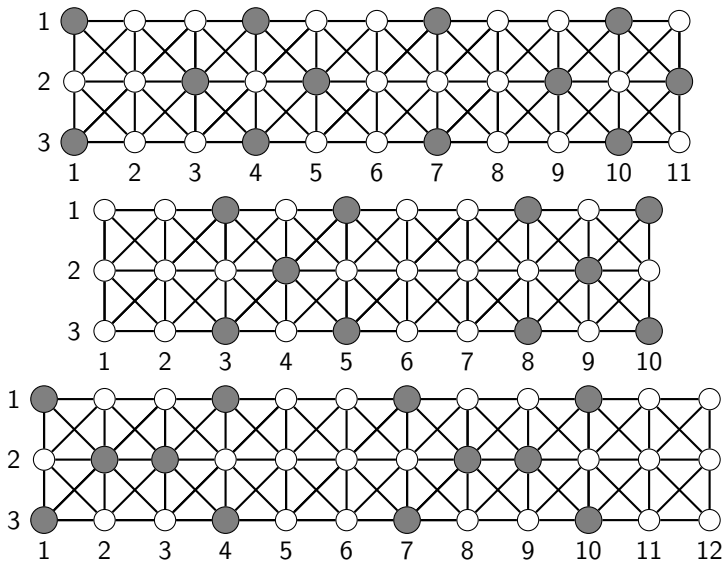


Figura: Id code of King grid with 3 rows and density 1/3

King grid with 3 rows

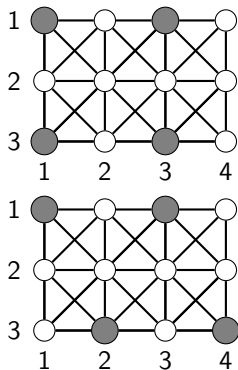


Figura: Id code of King grid with 3 rows and density $1/3$

Proof of King grid with 3 rows

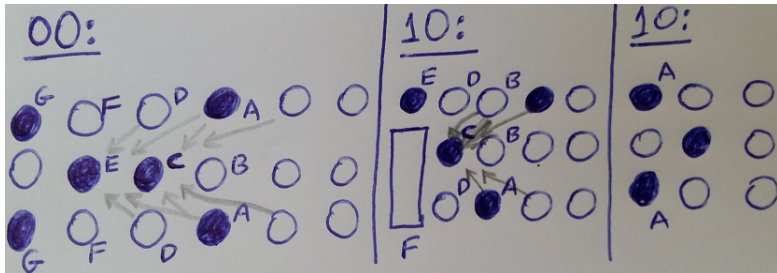


Figura: Proof for king grid with 3 rows

King grid with 4 rows

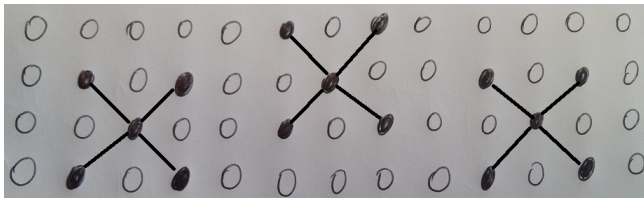


Figura: Id code of King grid with 4 rows and density $5/16$

Proof of King grid with 4 rows

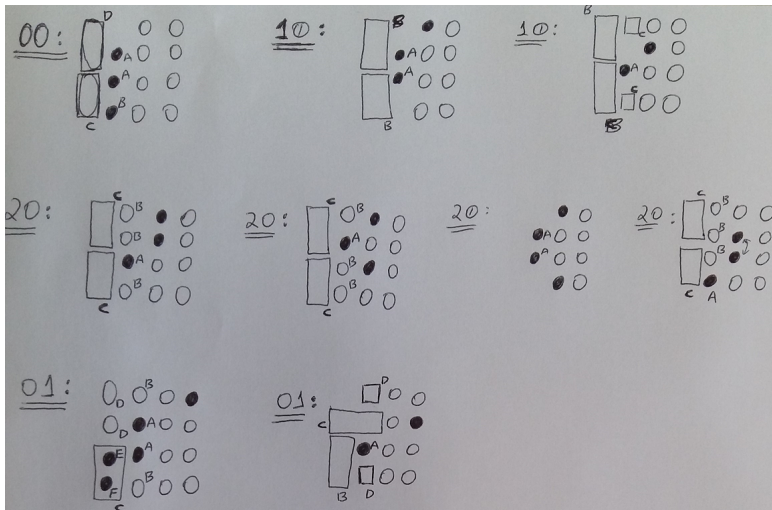


Figura: Proof for king grid with 4 rows - case 1

Proof of King grid with 4 rows

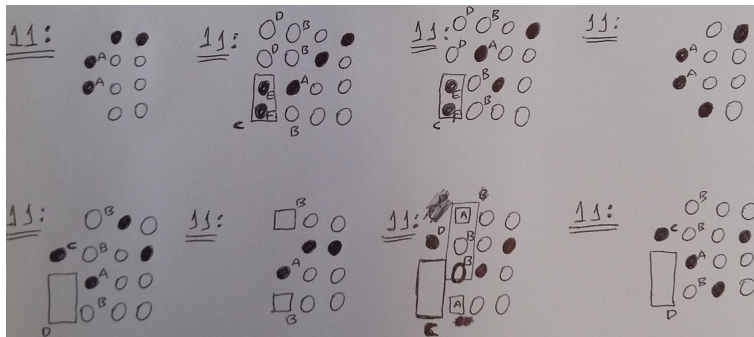


Figura: Proof for king grid with 4 rows - case 2

King grid with 5-6 rows

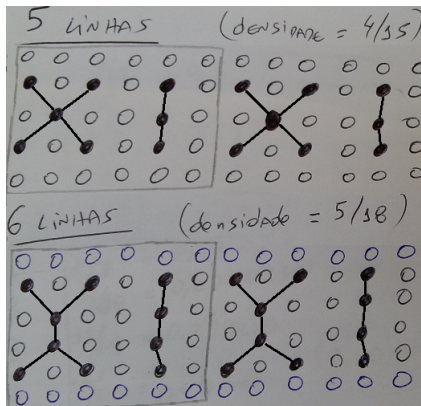


Figura: Optimum Id codes of King grid with 5-6 rows

King grid with 5 rows

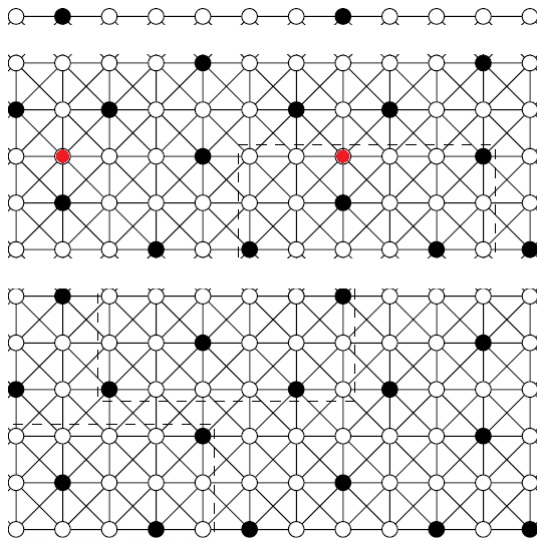


Figura: Optimum Id code of King grid with 5 rows and density 4/15

King grid with 6 rows

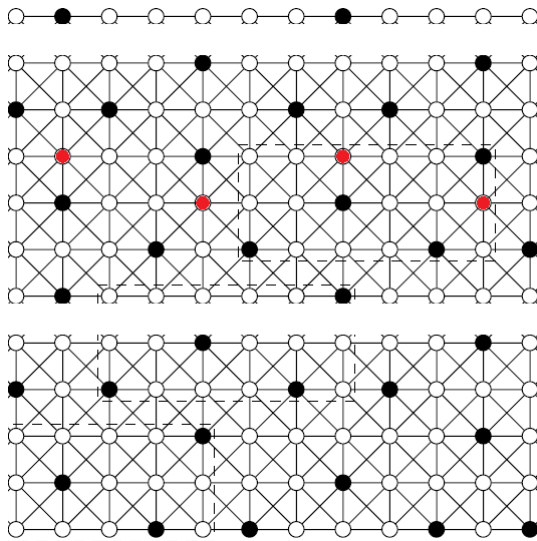


Figura: Optimum Id code of King grid with 6 rows and density 5/18

King grid with 7 rows

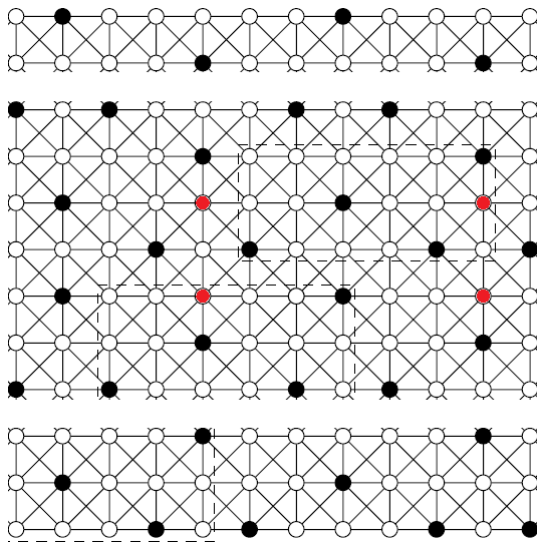


Figura: Id code of King grid with 7 rows

King grid with 8 rows

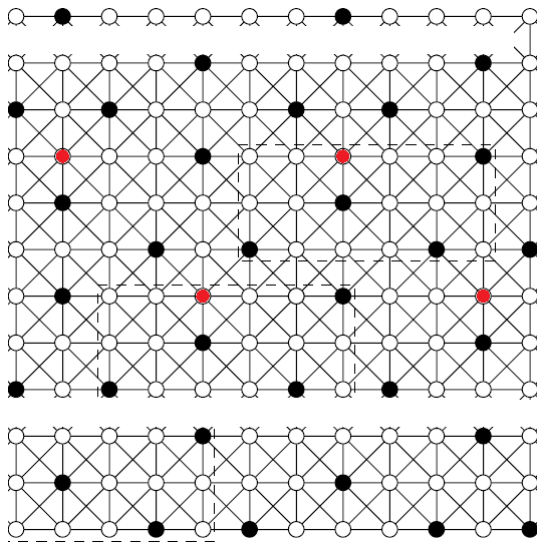


Figura: Id code of King grid with 8 rows

King grid with 9 rows

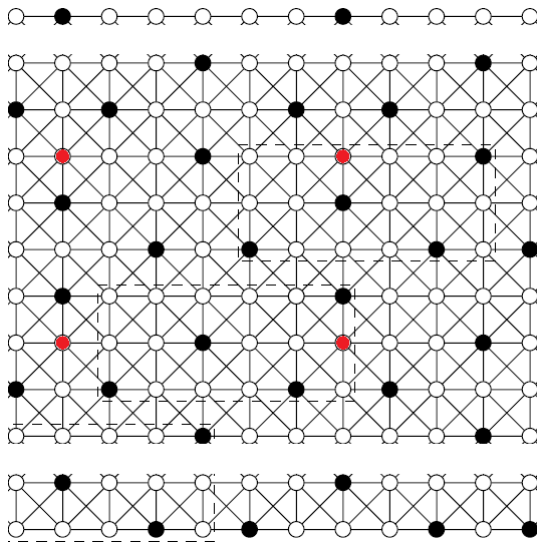


Figura: Id code of King grid with 9 rows

King grid with 10 rows

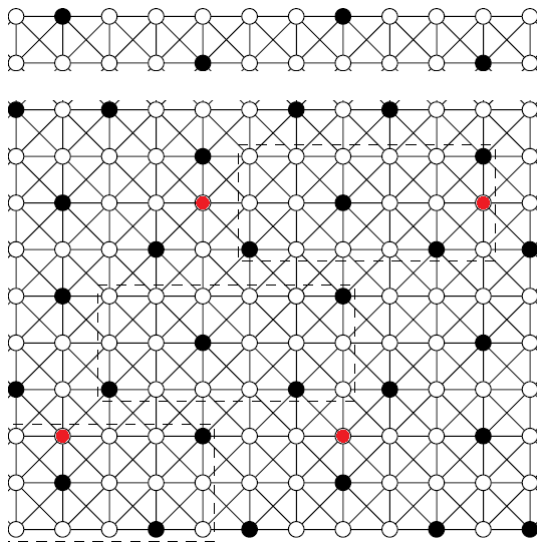


Figura: Id code of King grid with 10 rows

King grid with 11 rows

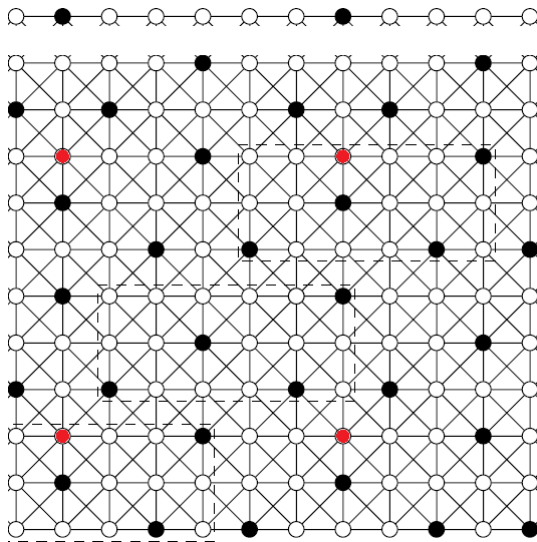


Figura: Id code of King grid with 11 rows

Results on Hexagonal grids with k rows

[Rudini, Gabriel, Yoshiko, 2023]:

- ▶ Hexagonal Grid H_k with k rows
- ▶ $d(H_1) = 1/2 = 0.5$
- ▶ $d(H_2) = 9/20 = 0.45\dots$
- ▶ $d(H_3) = 6/13 = 0.4615\dots$
- ▶ $d(H_4) = 7/16 = 0.4375$
- ▶ $d(H_5) = 11/25 = 0.44$
- ▶ $d(H_6) \leq 7/16 = 0.4375$
- ▶ $d(H_n) \leq 3/7 + \frac{r}{14 \cdot n}$, where $r = n \bmod 7$

Conjectura [Cohen et al., 2000]: $d(\mathcal{G}_H) = 3/7 = 0.42857$

Hexagonal grid with 2 rows - density $9/20 = 0.45$

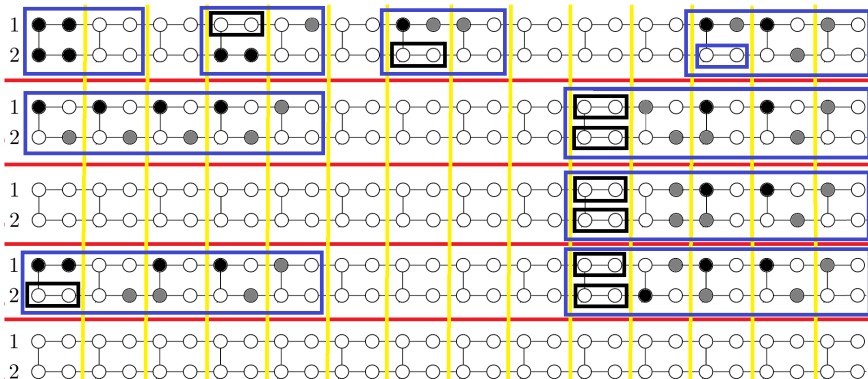


Esse aqui em cima é o código ótimo do H2 (hexagonal com 2 linhas), com densidade 0.45

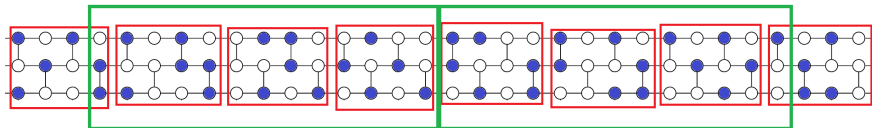
Ideia da prova: Considerar conjuntos C em sequência como esse:



Queremos provar densidade do código ótimo = 0.45. Portanto, dado um código qualquer, a carga de um conjunto C é seu número de vértices. Para densidade 0.45, cada C deve ter carga $0.45 \cdot 4 = 1.8$ no final. Vamos garantir que C com carga 0 ou 1 receba carga de algum C com carga sobrando. A descarga será a seguinte: um C com carga acima de 1.8 doará sua sobra apenas para a direita, para algum C que esteja precisando. Todos os casos são considerados abaixo. Vértices cinzas são suposições. Vértices pretos são conclusões.



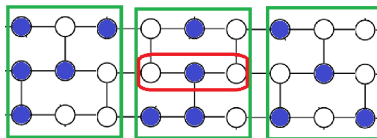
Hexagonal grid with 3 rows - density $6/13 = 0.4615\dots$



Código único do H_3 com densidade $6/13$: computer assisted proof.

Técnica: Grafo de configurações de 4 colunas. Cada configuração possível é um vértice. Total de vértices = $2^{4 \cdot 3} = 4096$ vértices. Dois vértices são adjacentes se as colunas internas estão bem identificadas. O peso da aresta é a média do número de vértices.

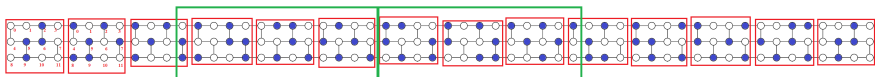
Todo código repetitivo é representado por um ciclo e, para todo código de H_k , existe um código repetitivo com densidade menor ou igual.



ERRO

Hexagonal grid with 3 rows - density $6/13 = 0.4615\dots$

$k = 3$ linhas $\Rightarrow 2^{4 \cdot 3} = 4096$ configurações/vértices



Código único do H_3 com densidade $6/13$: computer assisted proof.

Ideia 1: Problema Minimum Mean Cycle: Obter um ciclo cuja média dos pesos das arestas é mínima. Polinomial: Algoritmo de [\[Karp, 1978\]](#) com tempo $O(n \cdot m)$.

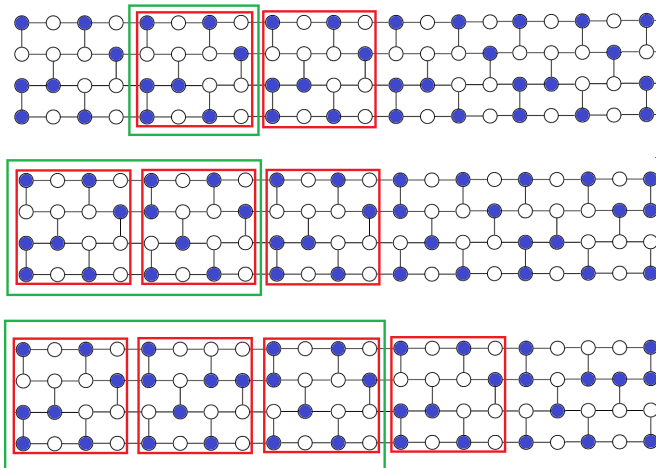
Ideia 2: Subtrair do peso de cada aresta o valor $4 * 3 * D$, onde D é a densidade que se quer provar mínima. Aplica-se Floyd para obter o ciclo de peso mínimo. Se o valor for exatamente zero, D é a densidade mínima.

Justificativa: Método da Descarga. $12 * D$ é a carga desejada.

Configurações com sobra doam para a direita no código. Se o peso mínimo é negativo, há um código em que a descarga falhou. Se o peso mínimo é positivo, todo código tem sobra de carga.

Hexagonal grid with 4 rows - density $7/16 = 0.4375$

$k = 4$ linhas $\Rightarrow 2^{4 \cdot 4} = 2^{16} = 65536$ configurações.

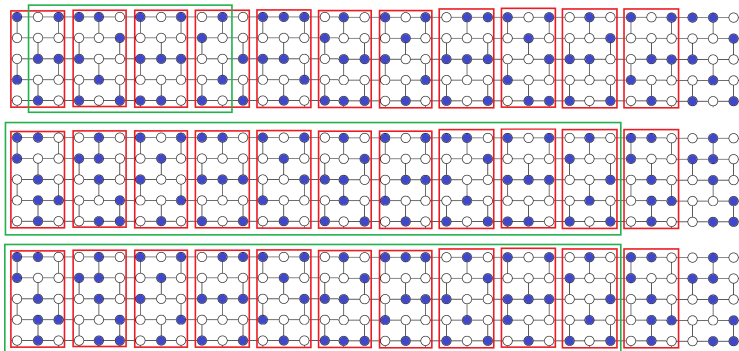


Hexagonal grid with 5 rows - density $11/25 = 0.44$

$k = 5$ linhas $\Rightarrow 2^{4 \cdot 5} = 2^{20} = 1.048.576$ configurações
(tempo absurdo e estouro de memória)

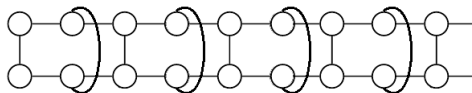
Solução: Configurações de 3 colunas: $2^{3 \cdot 5} = 32768$ configurações

Problema: O algoritmo pode retornar códigos inválidos. No entanto, se retornar um código com peso zero que é válido, então o código é mínimo.



Results on Hexagonal torus with k rows

- ▶ $d(H_2^*) = d(S_2) = 3/7$ [Daniel, Gravier, Moncel, 2004]

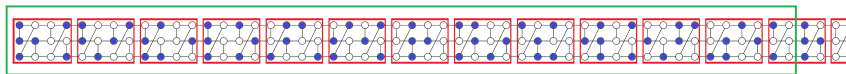


[Rudini, Gabriel, Yoshiko, 2023]: Hexagonal Torus H_k^* with k rows

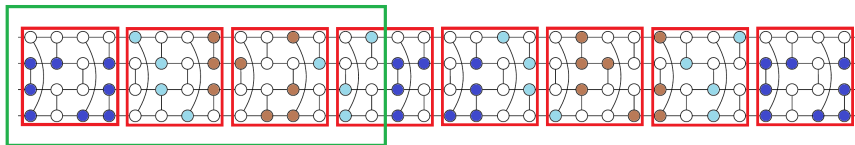
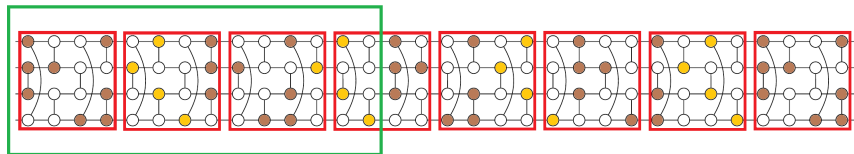
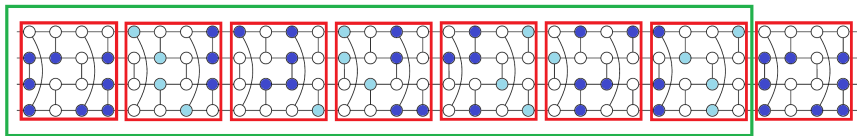
- ▶ $d(H_3^*) = 11/25 = 0.44$
- ▶ $d(H_4^*) = 3/7$
- ▶ $d(H_5^*) = 3/7$
- ▶ $d(H_6^*) \leq 3/7$
- ▶ $d(H_n) \leq 3/7 + \frac{r}{14 \cdot n}$, where $r = n \bmod 7$

Conjectura [Cohen et al., 2000]: $d(\mathcal{G}_H) = 3/7 = 0.42857$

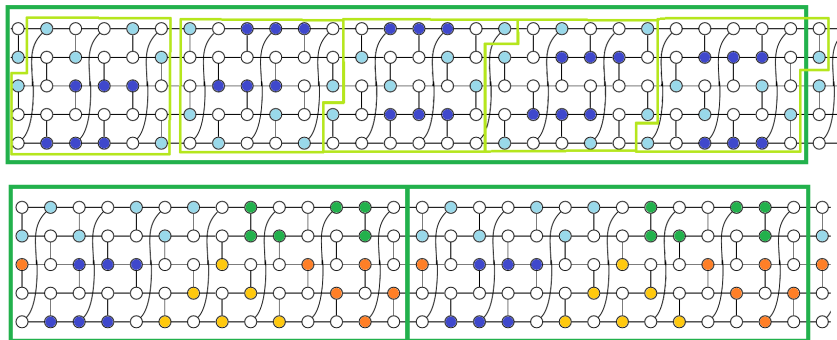
Hexagonal torus with 3 rows - density $11/25 = 0.44$



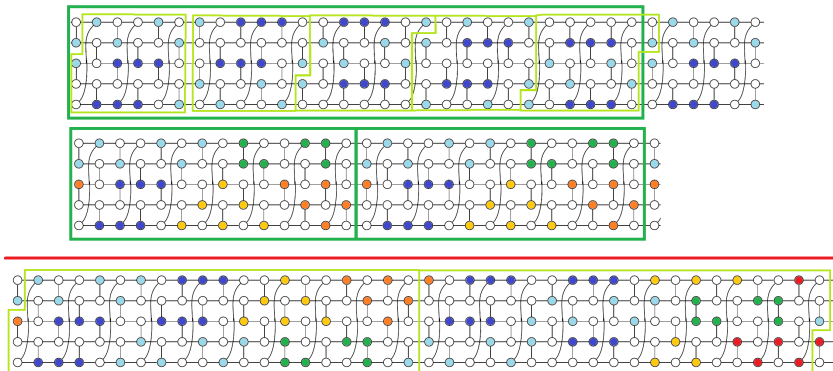
Hexagonal torus with 4 rows - density $3/7 = 0.42857$



Hexagonal torus with 5 rows - density $3/7 = 0.42857$

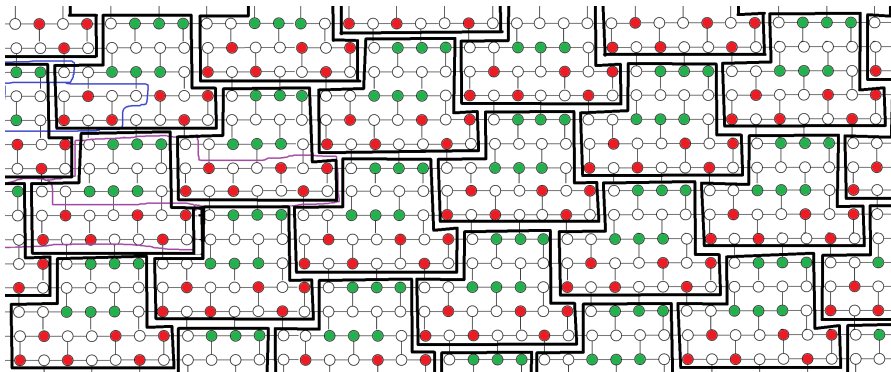


Hexagonal torus with 5 rows - density $3/7 = 0.42857$



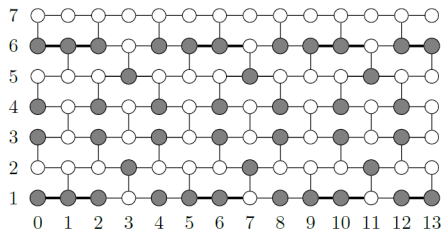
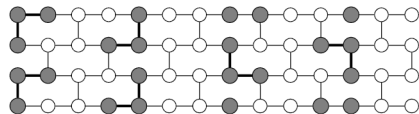
Infinite idcodes in the Hexagonal grid with density $3/7$

Primeiro código com densidade $3/7$ no grid hexagonal infinito com padrão de 5 linhas.



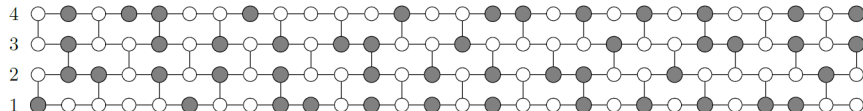
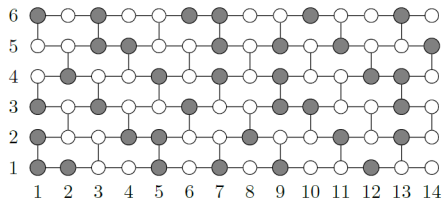
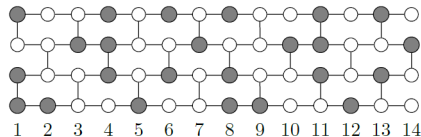
Infinite idcodes in the Hexagonal grid with density $3/7$

[Cohen et al., 2000]



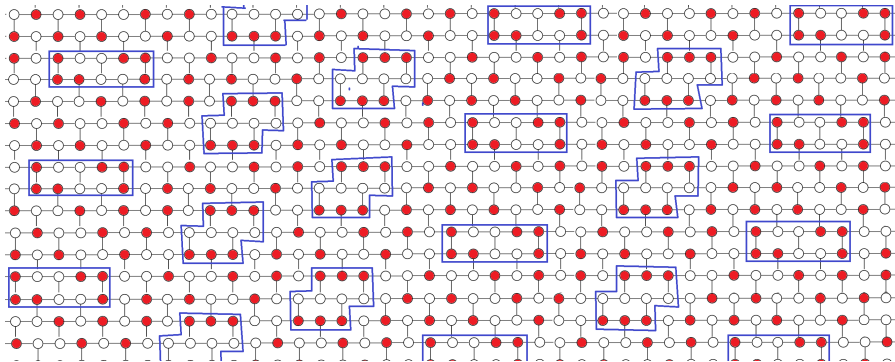
Infinite idcodes in the Hexagonal grid with density 3/7

[Cukierman et al. 2013]



Infinite idcodes in the Hexagonal grid with density $3/4$

Infinitos códigos do grid infinito combinando os códigos ótimos de H_5^* .



Também é possível combinando os códigos ótimos de H_4^* .

Também é possível combinando os códigos ótimos de H_4^* e H_5^*

Mas vocês já estão cansados de tantas figuras... Vou encerrar aqui com essa linda figura, que me deu muito trabalho !! Obrigado