Hardness of general position games

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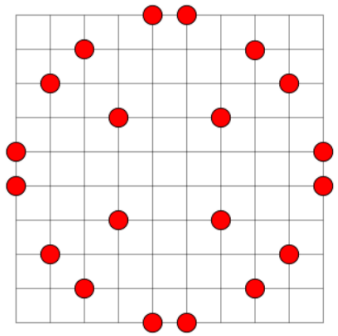
LAWCG-2022, Curitiba, October 19, 11h30
Celebrating Jayme's 80th birthday

The General Position Problem

- Points in general position: no three points in the same line.



Maximum General Position Set
in the 10 x 10 grid:

20 points
(2 per line and 2 per column)

- Dudeney's "Puzzle with Pawns" in the book "Amusements in Mathematics" of 1917.
- "No-Three-In-Line Problem": Given n, find the maximum number of points in general position in the $n \times n$ grid.
- CS variation: given points in the plane $n \times n$, find the maximum number of points in general position.
- Graph variation: given a graph, select the maximum number of vertices in general position: no selected vertex is in the shortest path between other two selected vertices.

General Position versus Geodesic Convexity

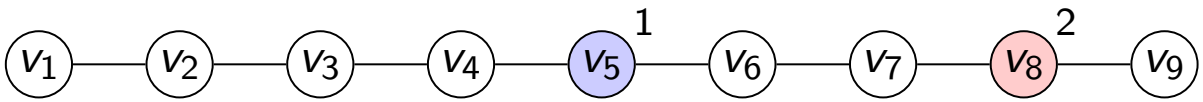
- Geodesic closure $I[S]$ of S: S and all vertices in shortest paths between two vertices of S.
- Equivalent to the Interval Set of the geodesic convexity.

- General Position Problem: given a graph G, select the max number of vertices in general position: No selected vertex is in a geodesic between other two selected vertices
- Max Geodesic Convex Set: given G, select the maximum number of vertices such that: No **non**-selected vertex is in a geodesic between two selected vertices

- Classical problems: There are many papers investigating them

Game Variants

- Buckley-Harary-1985' geodesic game:
  two players (A/B) alternately select vertices which are not in the geodesic closure of the vertices selected so far.



- Klavžar-Neethu-Chandran-2021' general position game:
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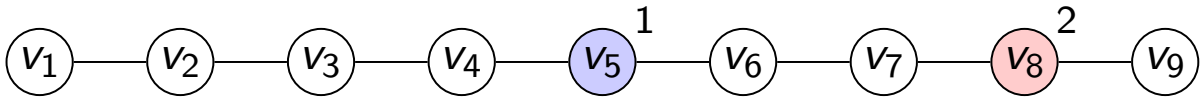

- **Normal variant:** The last to play wins (achievement game)
- **Misèrè variant:** The last to play loses (avoidance game)

- Zermelo'1913: One of the players has a winning strategy
- Problem: Given a graph, Player A has a winning strategy?

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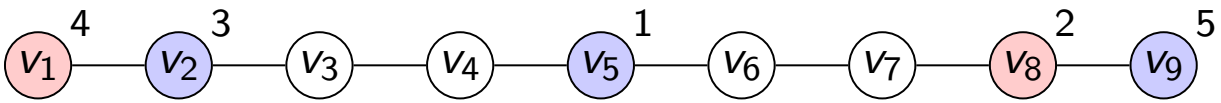

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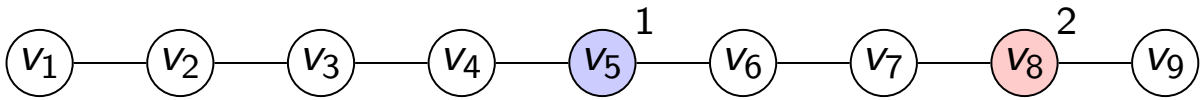

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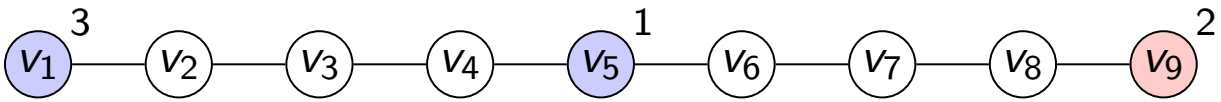

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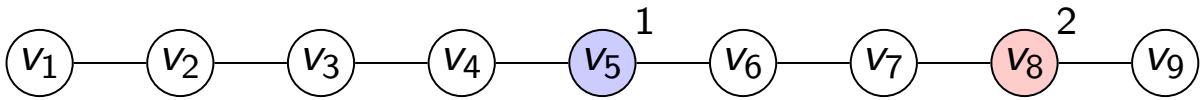

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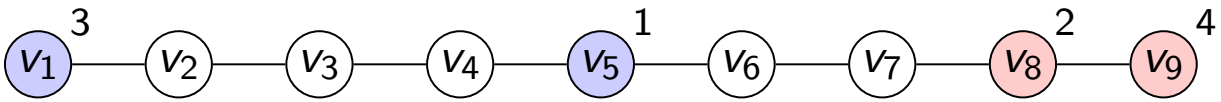

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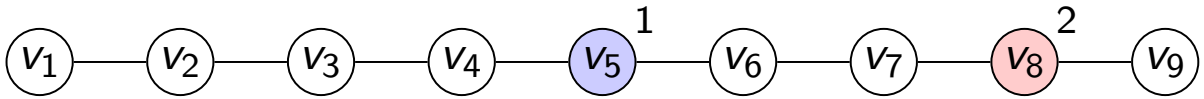

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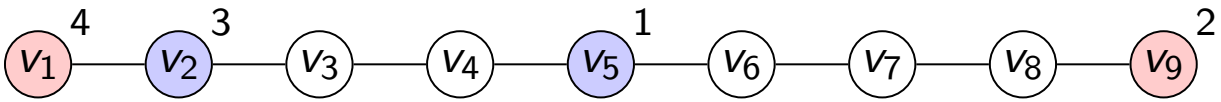

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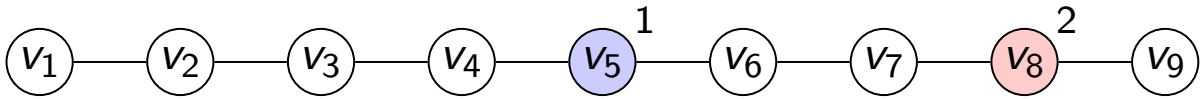

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Known results on the game problems

Geodesic games

- [Buckley-Harary-1985]: Solved for some graph classes.
- [Nečásková-1988]: Solved for wheel graphs.
- [Haynes-Henning-Tiller-2003]: Trees and complete multipartite.

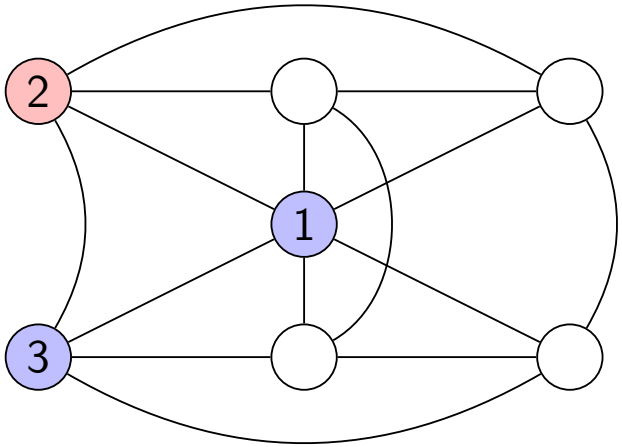
General Position games

- [Chandran-Klavžar-Neethu-2021]: achievement game on Hamming graphs, Cartesian and lexicographic products.

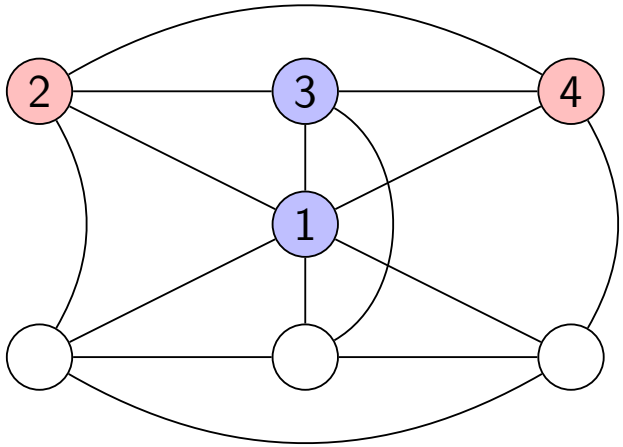
Our results

- We prove that the 4 games are PSPACE-Complete even in graphs with diameter at most 4:
  - 1985' achievement geodesic game
  - 1985' avoidance geodesic game
  - 2021' achievement general position game
  - 2021' avoidance general position game
- For this, we had to prove that the misèrè versions of the 1978'Node-Kayles game and the 1978'Clique-Forming game are PSPACE-Complete: games of obtaining an indep. set or a clique, resp., where the loser is the last to pick a vertex.
- Polytime algorithms of gp-avoidance game in rook's graphs, grids, cylinders, and lexicographic products with complete second factors.

Examples



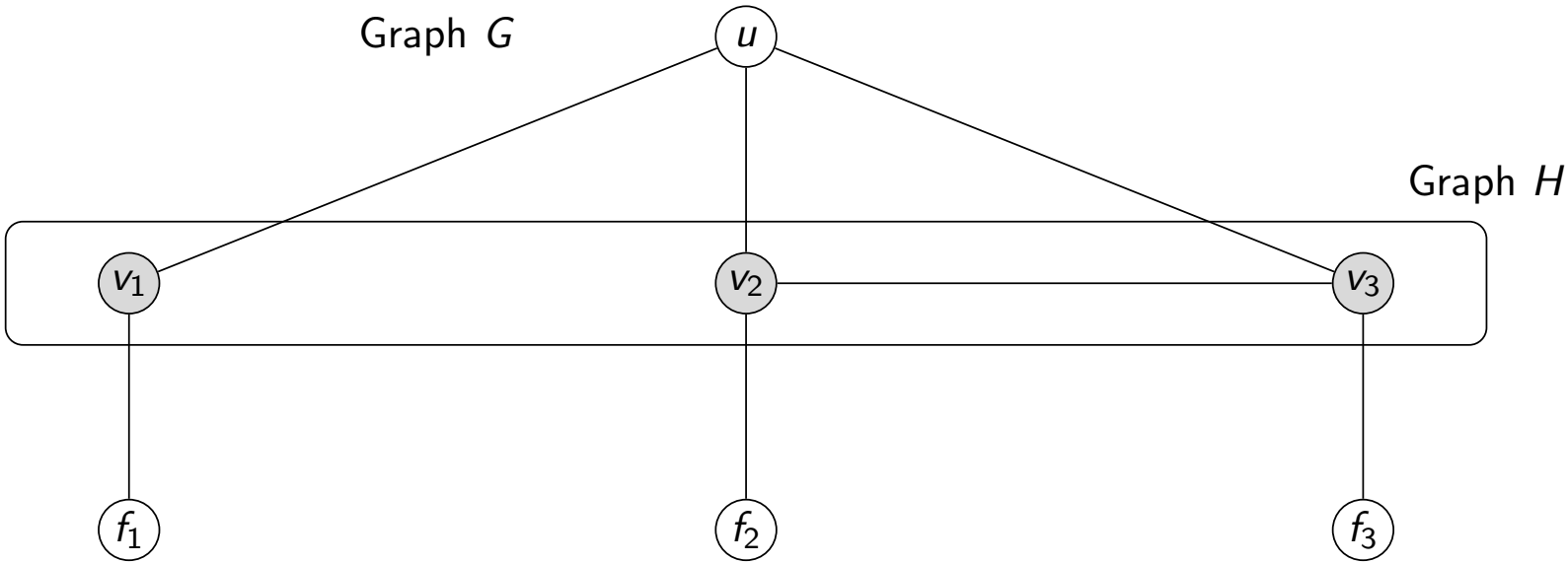
Player A wins the normal gp and clique-forming games



Player A wins the misèrè gp and clique-forming games

gp-achievement game is PSPACE-Complete

Reduction from the Clique-Forming game: diameter 4 graph.



Player A has a winning strategy in the gp-achievement game
if and only if
Player B has a winning strategy in the Clique-forming game

gp-avoidance game is PSPACE-Complete

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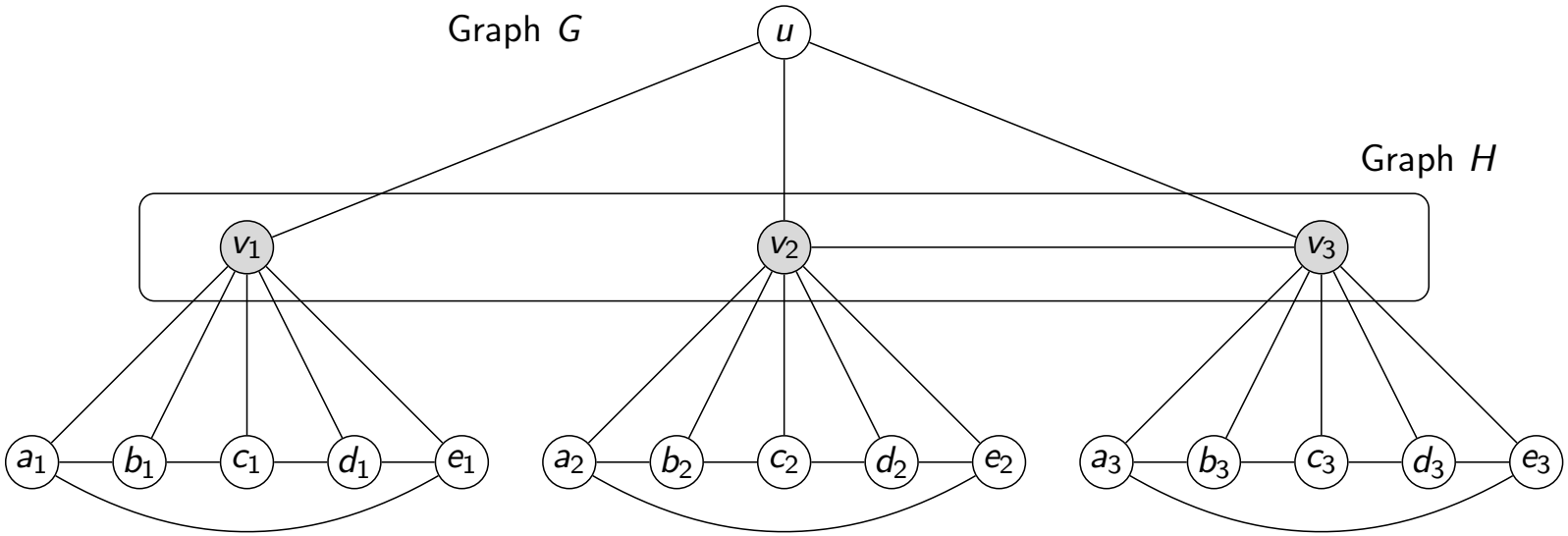

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Rápida homenagem ao Jayme

- Jayme apoiou nosso grupo pesquisa da UFC, Fortaleza, Brazil
- PARGO Research Group - Parallelism, Graphs & Optimization
- Jayme nos visitou em várias ocasiões: defesas de mestrado, defesas de doutorado, palestras
- Suas palestras sempre tinham um convite para trabalhar nos assuntos apresentados
- Um exemplo são os Problemas de Convexidade em Grafos
- Pelo menos 15 artigos do PARGO em periódicos nesse tema
- Prazer de ter artigos com ele - Sou Jayme número 1

- 2012 - V.Campos, A.Silva, R.Sampaio, J.L.Szwarcfiter. Graphs with few P4's under the Convexity of paths or order three. CTW-2012. 70 anos
- 2015 - V.Campos, R.Sampaio, A.Silva, J.L.Szwarcfiter. Graphs with few P4s under the complexity of paths of order three. Discrete Applied Math
- 2017 - M. Kanté, R. Sampaio, V.F.Santos, J.L.Szwarcfiter. On the geodesic rank of a graph. Journal of Combinatorics
- 2018 - R. Araújo, R. Sampaio, V.F.Santos, J.L.Szwarcfiter. The convexity of induced paths of order three and applications: Complexity aspects. Discrete Applied Math.

Thank you, Jayme !!

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