Hardness of variants of the graph coloring game

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Introduction

Proper coloring

- The vertices of graph are colored
- Two adjacent vertices must receive distinct colors
- $\chi(G)$: chromatic number (min number in a proper coloring)

Greedy coloring

- Proper vertex coloring / colors are integers
- Take an ordering of the vertices.
- A vertex must receive the minimum available color.
- $\Gamma(G)$: Grundy number (max number in a greedy coloring)

$$\Gamma(G) \geq \chi(G)$$

Graph coloring game and Greedy coloring game

- **Instance:** a graph G and a set C of colors/integers
- Two players **Alice** and **Bob** alternate their turns in choosing an uncolored vertex to be **proper colored** by an integer of C
- Alice wins if all vertices are successfully colored
- Zermelo-von Neumann Th.: finite perfect-information game without draw: Alice or Bob has a winning strat

Graph coloring game g_A $(\chi_g(G) \geq \chi(G))$

- Alice **starts**. They may use **any** possible integer of C
- **Game chromatic** number $\chi_g(G)$: minimum number of colors s.t. Alice has a winning strategy in the graph coloring game

Greedy coloring game g_A^* $(\chi(G) \leq \Gamma_g(G) \leq \Gamma(G))$

- Alice starts. They must use **the min.** possible integer of C
- **Game Grundy number** $\Gamma_g(G)$: minimum number of colors s.t. Alice has a winning strategy in the greedy coloring game

Variants of these coloring games

Graph YZ-coloring game $g_{YZ}(\chi_g^{YZ}(G) \geq \chi(G))$

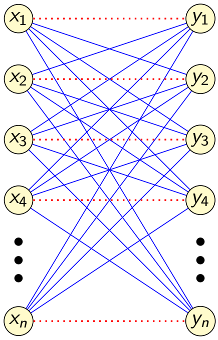
- $Y \in \{A, B\}$ and $Z \in \{A, B, \textit{no one}\}$
- Y starts the game and Z may pass turns
- Alice and Bob may use **any** possible integer of C
- *YZ*-**game chromatic** number $\chi_g^{YZ}(G)$: min number of colors s.t. Alice has a winning strategy in the YZ-coloring game
- We omit Z when it is "no one": $\chi_g^A(G) = \chi_g(G)$ is the original game chromatic number.

Greedy YZ-coloring game g_{YZ}^* $(\chi(G) \leq \Gamma_g(G) \leq \Gamma(G))$

- Same idea, but they must use **the min.** possible integer of C
- *YZ*-**game Grundy number** $\Gamma_g^{YZ}(G)$: min number of colors s.t. Alice has a winning strat in the greedy YZ-coloring game
- We omit Z when it is "no one": $\Gamma_g^A(G) = \Gamma_g(G)$ is the original game Grundy number.

An example for both coloring games

- Complete bipartite graph without a matching
- **If Alice is the first** to play, Bob can force n colors: just play in the non-neighbor of Alice's last vertex.
- **If Bob is the first** to play, Alice wins with 2 colors.
  Normal game: Alice colors non-neighbor with other color.
  Greedy game: color same side of Bob's first vertex.



Literature - χ_g

Graph coloring game / Game chromatic number $\chi_g(G)$

- First mentioned by [Brams] and described by [Gardner'81]
- [Bodlaender'91] reinvented as "Color Construction Game"

- forest ≤ 4 [Faigle...'93], outerplanar ≤ 7 [Kierstead...'94]
- $\chi_g \leq (\chi_a + 1)^2$ acyclic chromatic number χ_a [Dinski,Zhu'99]
- $\chi_g(P_k) \leq 3k + 2$ for partial k trees [Zhu'00]
- $\chi_g(G) \leq 5$ in cacti [Sidorowicz'07]
- Asympt. behavior $\chi_g(G(n, p))$ [Bohman, Frieze, Sudakov'08]
- Planar graphs: $\chi_g \leq 17$ [Zhu'08], $\chi_g \leq 13$ [Sekiguchi'14, girth$\geq$ 4], $\chi_g \leq 5$ [Nakprasit'18, girth$\geq$ 7]

Literature - χ_g - Complexity

- [Bodlaender'91]: "*The complexity of Color Construction Game is an interesting open problem*"
- [Dunn et. al'15]:
  "*more than two decades later,* ***this question*** *remains open*".
- [Andres,Lock'19]: Introduced the variants χ_g^{YZ}. "*The question of PSPACE-hardness remains open for all the game variants mentioned above*", including the original one.
- [Costa,Soares,**Sampaio**'19]: Proved that $\chi_g(G)$ is PSPACE-hard, solving Bodlaender's 30-years question.

Literature - Γ_g

Greedy coloring game / Game Grundy number $\Gamma_g(G)$

- Introduced by [Havet, Zhu'13]
- $\Gamma_g(G) = \chi(G)$ in cographs [Havet,Zhu'13]
- $\Gamma_g(F) \leq 3$ in forests [Havet, Zhu'13]
- $\chi_g(G) \leq 7$ in partial 2-trees [Havet, Zhu'13]
- Two questions of [Havet,Zhu'13]
  - (*) $\chi_g(G)$ is upper bounded by a function of $\Gamma_g(G)$?
  - (**) $\Gamma_g(G) \leq \chi_g(G)$ for every graph G?
- (*) = NO [Krawczyk,Walczak'15]
- (**) is still open
- [Costa,Soares,**Sampaio**'19]: $\Gamma_g(G)$ is PSPACE-hard.

Connected graph coloring game

Connected game chromatic number ($\chi_{cg}(G) \geq \chi(G)$)

- Similar to the original graph coloring game: Alice starts and no one may pass turns
- But colored vertices must induce a connected subgraph
- **Connected game chromatic** number $\chi_{cg}(G)$: min number of colors s.t. Alice has a winning strategy in the connected graph coloring game

Literature - $\chi_{cg}(G)$

- Introduced by [Charpentier,Hocquard,Sopena,Zhu'19]
- [CHSZ'19]: Alice wins with 2 colors in bipartite graphs
- [CHSZ'19]: Alice wins with 5 colors in outerplanar graphs
- [Bradshaw'20]: There are outerplanar 2-trees with $\chi_{cg}(G) = 5$

Our results

Complexity results

- Game chromatic n's χ_g^{YZ} are PSPACE-hard for all variants
- Game Grundy numbers Γ_g^{YZ} are PSPACE-hard for all variants
- Connected game chromatic number χ_{cg} is PSPACE-hard
- All the related decision problems are PSPACE-Complete, even if the number of colors is the chromatic number

$\chi_g^B(G)$ is PSPACE-hard - Bob starts

Zhu'99 open question: Graph coloring game "*exhibits some strange properties*". Does Alice have a winning strategy with $k+1$ colors if she has one with k colors?

We define three decision problems for the graph coloring game:

- (Problem g_B-1) Given G and k: $\chi_g^B(G) \leq k$?
- (Problem g_B-2) Given G and k: Does Alice have a winning strategy with k colors?
- (Problem g_B-3) Given G and $\chi(G)$: $\chi_g^B(G) = \chi(G)$?

Problems g_B-1 and g_B-2 are generalizations of Problem g_B-3, when we know $\chi(G)$ - just take $k = \chi(G)$

Problem g_B-3 is PSPACE-hard $\rightarrow$ g_B-1 and g_B-2 are PSPACE-hard

χ_g^B: Reduction from POS-CNF$_B$

CNF formula, only positive literals, Bob and Alice alternate turns setting variables true or false. Alice wins if the formula is true. Bob wins if he selects (false) all variables of a clause.

Example

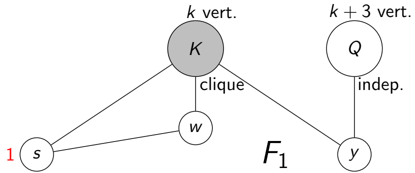
$(X_1 \vee X_2 \vee X_5) \wedge (X_1 \vee X_3 \vee X_5) \wedge (X_2 \vee X_4 \vee X_5) \wedge (X_3 \vee X_4 \vee X_5)$.
Bob has a winning strategy setting X_5 false first:

- X_1 True $\rightarrow X_4$ False;
- X_2 True $\rightarrow X_3$ False;
- X_4 True $\rightarrow X_1$ False;
- X_3 True $\rightarrow X_2$ False.

Good points

- POS-CNF$_B$ is PSPACE-Complete (POS-DNF [Shaefer'78])
- **Lemma:** If a player has a winning strategy in POS-CNF$_B$, the player also has a winning strat if the opponent pass turns.

χ_g^B: Important ingredient of the Reduction



Lemma:

Suppose that Bob colored vertex s in his first move. Alice has a winning strategy in F_1 with $k+2$ colors **iff** she colors vertex y first with the same color of s.

Proof:

$k+2$ colors **iff** (y and s) or (y and w) have the same color.

If Alice colors y with the same color of s, she wins.

If Alice colors y with other color, Bob wins by coloring w with a different color. Otherwise, Bob colors the vertices of Q with distinct colors (starting with the color of s). If one of this colors in Q is not in K, he colors w with this color, and he wins.

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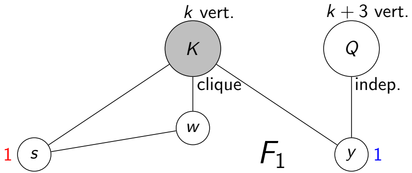

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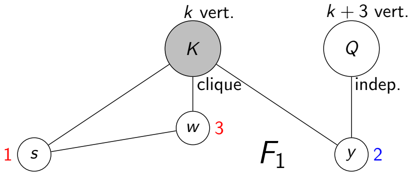

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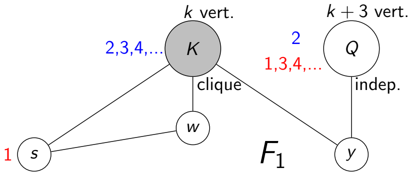

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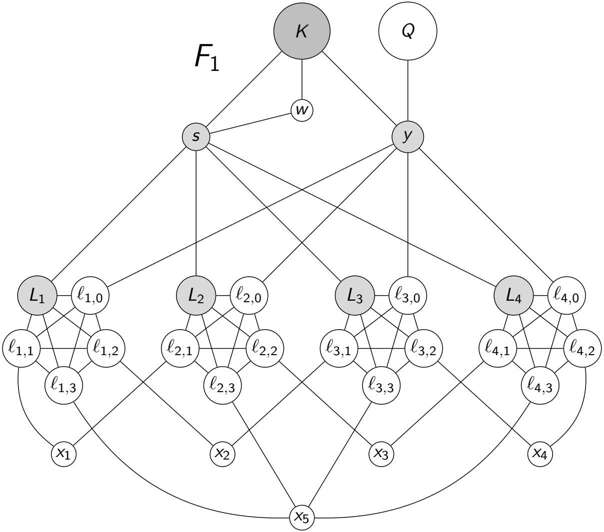
χ_g^B: Reduction from POS-CNF$_B$

$$(X_1 \vee X_2 \vee X_5) \;\wedge\; (X_1 \vee X_3 \vee X_5) \;\wedge\; (X_2 \vee X_4 \vee X_5) \;\wedge\; (X_3 \vee X_4 \vee X_5).$$



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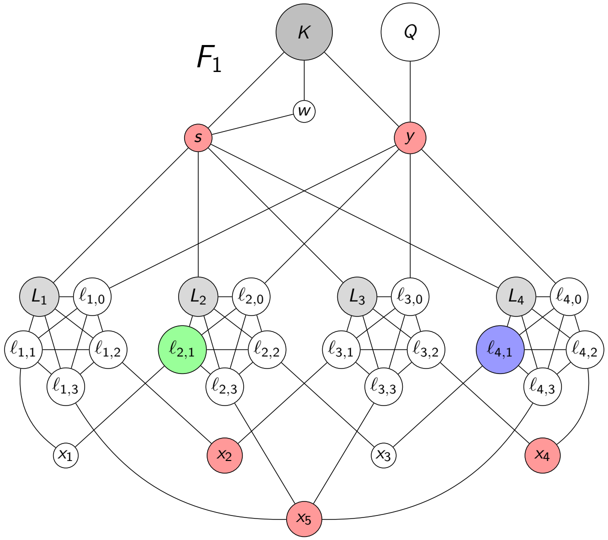


χ_g^{YZ} is PSPACE-hard for any Y, Z

Let $\chi_g^{A,A}(G)$, $\chi_g^{A,B}(G)$, $\chi_g^{B,A}(G)$ and $\chi_g^{B,B}(G)$: the minimum number of colors in C such that Alice has a winning strategy in $g_{A,A}$, $g_{A,B}$, $g_{B,A}$ and $g_{B,B}$, resp. For $Y, Z \in \{A, B\}$, let

- (Problem $g_{Y,Z}$-1) $\chi_g^{Y,Z}(G) \leq k$?
- (Problem $g_{Y,Z}$-2) Does Alice have a winning strategy in $g_{Y,Z}$ with k colors?
- (Problem $g_{Y,Z}$-3) $\chi_g^{Y,Z}(G) = \chi(G)$?

Corollary

For every $Y, Z \in \{A, B\}$, the decision problems $g_{Y,Z}$-1, $g_{Y,Z}$-2 and $g_{Y,Z}$-3 are PSPACE-complete.

Proof.

Reduce from POS-CNF or POS-CNF$_B$ if $Y = A$ or $Y = B$, resp.
Since a winning strategy in both problems implies a winning strat if the opponent pass turns, we are almost done...

□

χ_{cg} is PSPACE-hard - Connected

Three decision problems for the Connected graph coloring game:

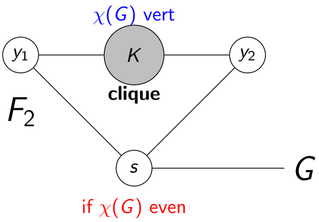
- (Problem cg$_A$-1) Given G and k: $\chi_{cg}(G) \leq k$?
- (Problem cg$_A$-2) Given G and k: Does Alice have a winning strategy with k colors?
- (Problem cg$_A$-3) Given G and $\chi(G)$: $\chi_{cg}(G) = \chi(G)$?

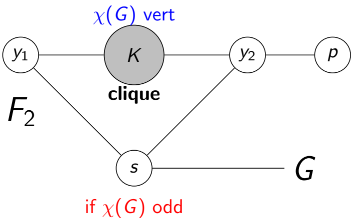
Problems cg$_A$-1 and cg$_A$-2 are generalizations of Problem cg$_A$-3, when we know $\chi(G)$ - just take $k = \chi(G)$

Problem cg$_A$-3 PSPACE-hard $\rightarrow$ cg$_A$-1 and cg$_A$-2 PSPACE-hard

Reduction from g$_B$-3: normal Coloring game when Bob starts.
• Given an instance $(G, \chi(G))$ of g$_B$-3,
produce an instance $(G', \chi(G'))$ of cg$_A$-3.

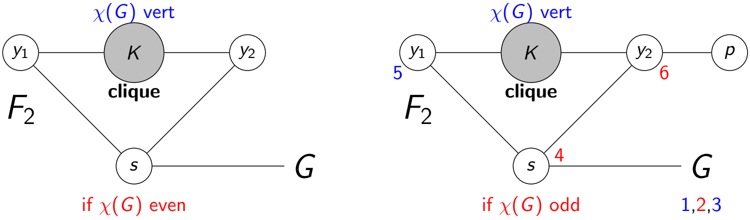
χ_{cg}: PSPACE-hard reduction from g_B-3





- We may assume $|V(G)|$ odd
- $\chi_{cg}(G') = k + 1 \;\Rightarrow y_1$ and y_2 have the same color
- Alice must color y_1, y_2 or K first, since $|V(G) \cup \{s\}|$ even.
- If Alice has a winning strategy in g_B-3, she colors y_1 first and can guarantee that Bob is the first to play in G
- If Bob has a winning strategy in g_B-3, he can guarantee he is the first to play in G

χ_{cg}: PSPACE-hard reduction from g_B-3



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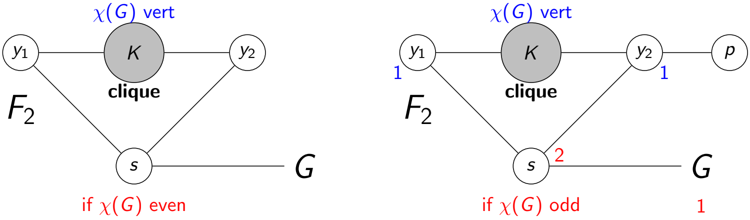
χ_{cg}: PSPACE-hard reduction from g_B-3



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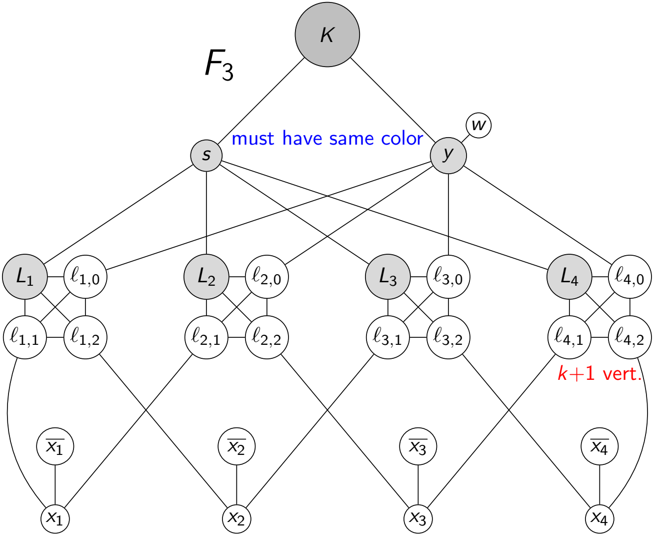
$\Gamma_g^B(G)$ is PSPACE-hard - Reduction POS-CNF

$(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_4) \wedge (X_3 \vee X_4)$



Conclusion

Our results

- Game chromatic n's χ_g^{YZ} are PSPACE-hard for all variants
- Game Grundy numbers Γ_g^{YZ} are PSPACE-hard for all variants
- Connected game chromatic number χ_{cg} is PSPACE-hard
- All the related decision problems are PSPACE-Complete, even if the number of colors is the chromatic number

THANK YOU !!