Target Set Selection with Maximum Activation Time

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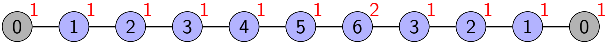
Target Set Selection Model - TSS Model (G, τ)

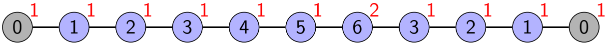


- **Instance:** Graph G, threshold funct $\tau : V(G) \to \{1, \ldots, \Delta\}$, where $\Delta = \Delta(G)$ is the max. degree of G
- $1 \leq \tau(v) \leq d(v)$, where $d(v)$ is the degree of v [Chen, 2009]

Activation process in a TSS Model (G, τ)

- **Instance:** TSS Model (G, τ) and $S_0 \subseteq V(G)$
- Vertices in S_0 are "active" and the others are "inactive".
- If v is inactive and has $\tau(v)$ active neighbors, v is activated.
- Irreversible process: active vertices remain active forever
- The process is synchronous: all inactive vertices update their status at the same time in each step of the process.
- $S_0, S_1, S_2, S_3, \ldots, S_t$: vertices activated at time $i = 0, 1, \ldots, t$
- If $S_0 \cup \ldots \cup S_t = V(G)$, we say that S_0 is a **target set** with **activation time** $t_\tau(S_0) = t$.

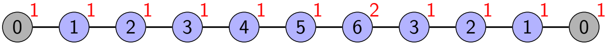
Target Set Selection Model



Activation process in a TSS Model (G, τ)

- S_0 is a **target set** if all vertices are active at the end.
- Thresholds are at least 1 and at most the degree of the vertex
- Introduced by **[Chen, 2009]**:
- **TSS-size problem**: find a target set with minimum size.
- Many recent works investigated the **TSS-size problem**.
- **[Chen, 2009]**: **TSS-size** is linear time solvable in trees
- **[Coelho et al., 2015]**: **TSS-size** is APX-hard

TSS-time problem



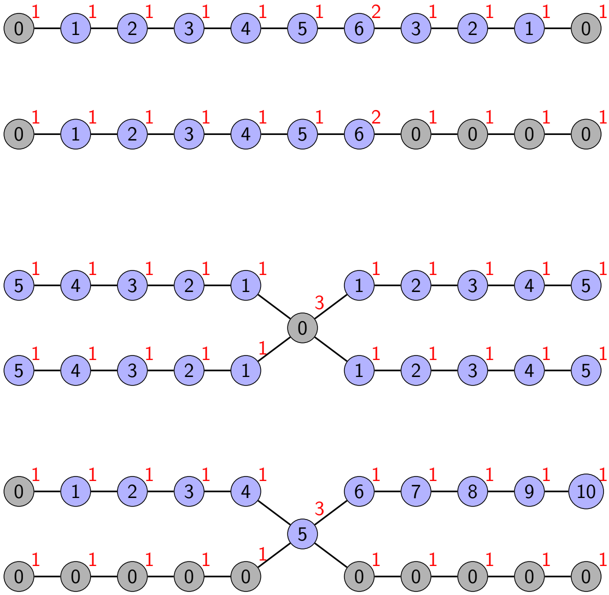
Activation process in a TSS Model (G, τ)

- TSS-size problem: find a target set with minimum size.
- **TSS-time problem**: find a target set with maximum time.

TSS-time decision problem

- **Instance:** Graph G, a threshold function τ in G and an integer $t \geq 0$.
- **Question:** Is there a target set $S_0 \subseteq V(G)$ with activation time at least t? That is, $t_\tau(S_0) \geq t$?

TSS-time problem



TSS-time problem

Activation process in a TSS Model (G, τ)

- TSS-size problem: find a target set with minimum size.
- **TSS-time problem**: find a target set with maximum time.

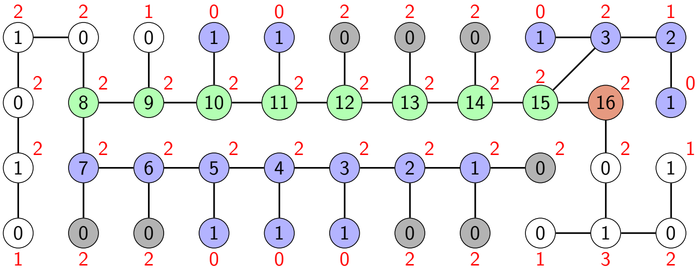
TSS-time results in the literature

- Question of Bollobás in the square grid with thresholds $=2$.
- r-neighbor bootstrap percolation: all thresholds $= r$.
- P_3-convexity: all thresholds are equal to $r = 2$ (hull set).
- [Przykucki, 2012]: exact value on the hypercube $2^{[n]}$
- [Benevides et al., 2013]: poly in the square grid
- [Marcilon et al., 2014]: poly for time $t = 3$ (in general graphs)
- [Marcilon et al., 2014]: poly for time $t = 4$ (bipartite graphs)

TSS-time problem

Activation process in a TSS Model (G, τ)

- TSS-size problem: find a target set with minimum size.
- **TSS-time problem**: find a target set with maximum time.
- **GTSS-time problem**: generalized TSS model (thresholds may be 0 or greater than the degree).
- Threshold 0: activated at time ≤ 1
- Threshold greater than the degree: must be in any target set



TSS-time problem

Activation process in a TSS Model (G, τ)

- TSS-size problem: find a target set with minimum size.
- **TSS-time problem**: find a target set with maximum time.
- **GTSS-time problem**: generalized TSS model (thresholds may be 0 or greater than the degree).

Known results - **GTSS-time** with $\tau(v) = 2, \ \forall v$

- [Marcilon et al., 2014]: NPC for time $t \geq 4$ (general graphs)
- [Marcilon et al., 2014]: NPC for time $t \geq 5$ (bipartite graphs)
- [Marcilon et al., 2018]: W[1]-hard on the treewidth
- Hardness results of GTSS-time do not apply directly to TSS-time, since most of these hardness reductions use many vertices of degree one.

Our results in the TSS-time problem

GTSS-time decision problem (G, τ, t)

- **Instance:** Graph G, generalized threshold func τ and $t \geq 0$.
- **Question:** Is there a target set S_0 with activation time $\geq t$?

Our results

- TSS/GTSS -time is linear/quadratic time solvable in trees
- Several hardness results (W[1]-hard, NP-hard in graph classes)
- TSS/GTSS-time in minor-closed graph class $\mathcal{C}$ are
  - FPT on the time t and τ^*, if $\mathcal{C}$ has bounded local-treewidth.
  - NP-complete for any fixed time $t \geq 4$ and $\tau^* = 2$, otherwise.
- where $\tau^* = \max_{v \in V(G)} \tau(v)$ is the maximum threshold.

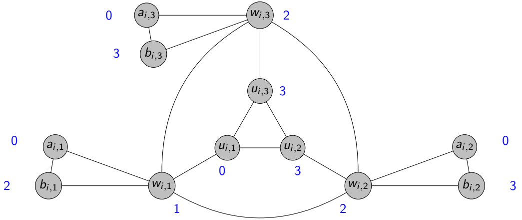
- [Ben-Zwi et al., 2011]: "treewidth governs the complexity of" **TSS-size** problem.
- Our conclusion: "Local-treewidth governs the complexity of" **TSS-time** problem.

Bounded local-treewidth graphs

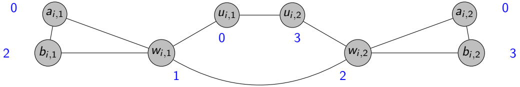
- Let $N_{\leq t}[v]$ be the t-neighborhood of v (distance $\leq t$ from v)
- G has bounded local-treewidth $\Longrightarrow G[N_{\leq t}[v]]$ has bounded treewidth for any fixed t and any vertex v
- $t_\tau(G) \geq t \;\Leftrightarrow\; \exists v, S_0(\text{target set}):\; t_\tau(v, S_0) \geq t$
- $\Leftrightarrow\; \exists v, S_0(\text{target set}):\; t_\tau(v, S_0 \cup N_{\geq t}[v]) \geq t$
- $\Leftrightarrow\; \exists v:\; t_\tau(G[N_{\leq t}[v]]) \geq t$
- For fixed t and τ^*, we obtain an MSO-formula for TSS-time $\Longrightarrow$ FPT parameterized by the treewidth
- $\Longrightarrow$ TSS-time is FPT in bounded local-treewidth graphs (parameterized by t and τ^*)

NP-hardness in apex graphs (thresholds = 2)

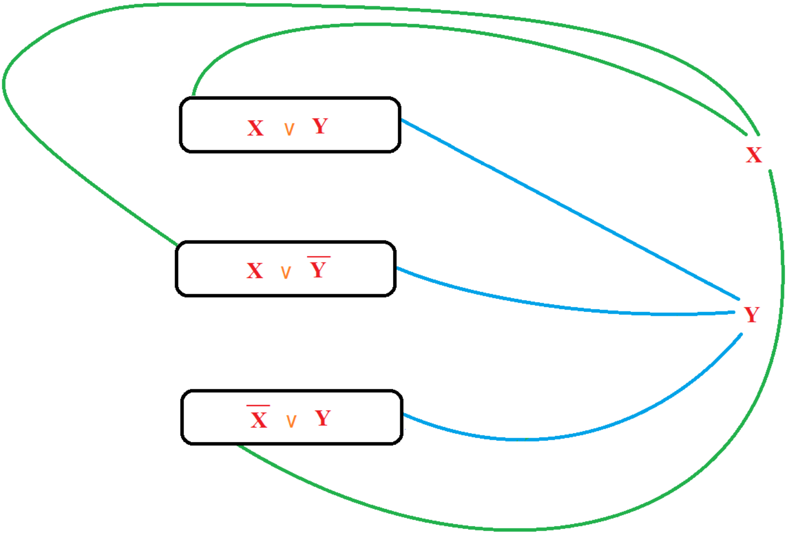
- Based on [Benevides, Campos, Dourado, Samp., Silva, 2015].
- Reduction from Restricted Planar 3-SAT: clauses with 2 or 3 literals, each positive literal appears twice, each negative literal appears once and the variable-clause graph is planar.
- For every clause with 3 literals, build the gadget below



- For every clause with 2 literals, build the gadget below



NP-hardness in apex graphs (thresholds $= 2$)



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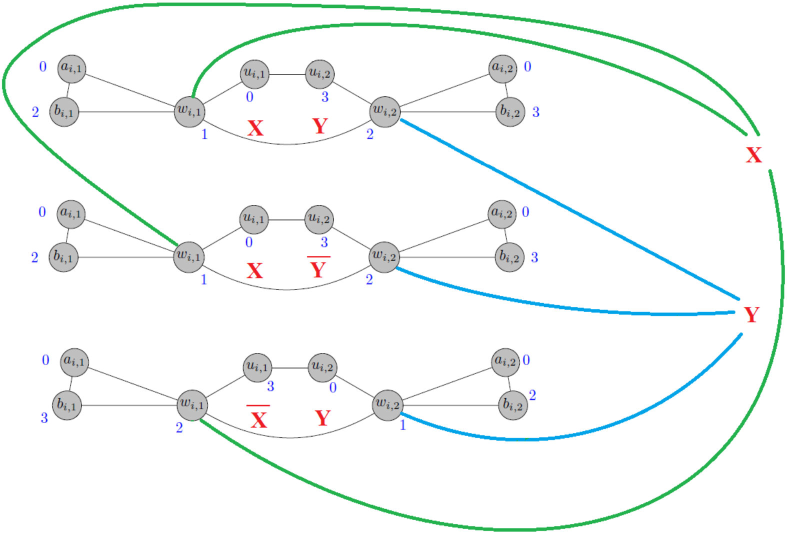

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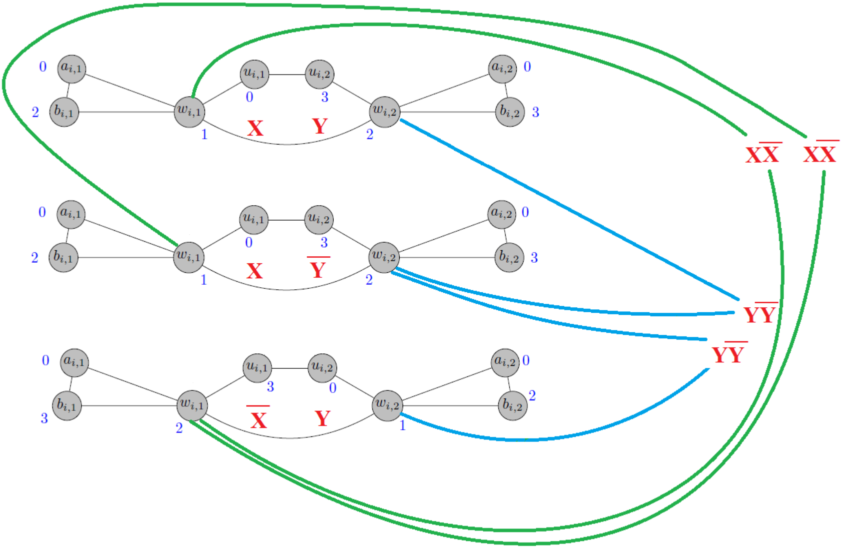

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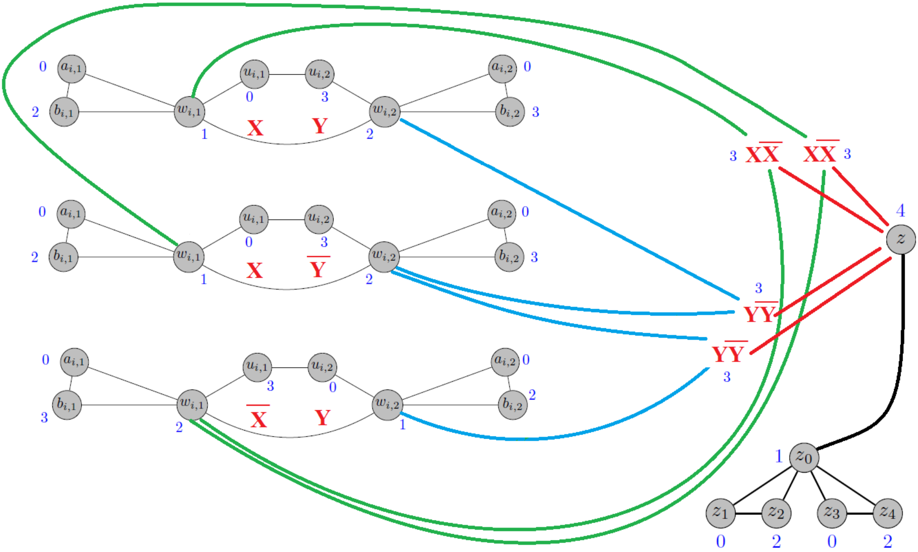


Local-treewidth "governs" TSS-time

Theorem

The TSS-time problem in a minor-closed graph class $\mathcal{C}$ is

- FPT on t and τ^*, if $\mathcal{C}$ has bounded local-treewidth.
- NP-complete for any fixed $t \geq 4$ and $\tau^* = 2$, otherwise.

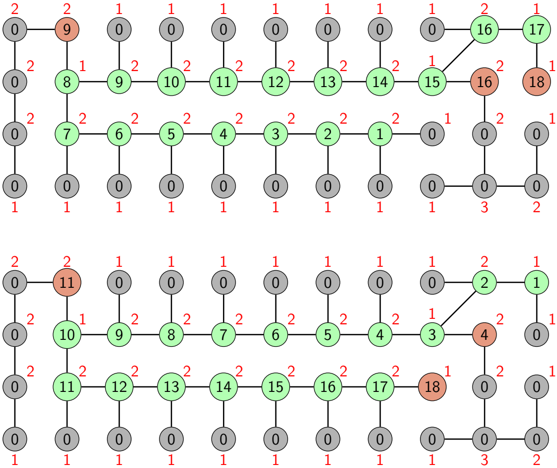
Proof:

[Eppstein,2000]: Let $\mathcal{C}$ be a minor-closed graph class. Then $\mathcal{C}$ has bounded local-treewidth if and only if $\mathcal{C}$ does not contain all apex graphs.

- Bounded local-treewidth $\Rightarrow$ FPT on t and τ^*
- Contains all apex graphs $\Rightarrow$ NP-complete for any fixed time $t \geq 4$ and $\tau^* = 2$

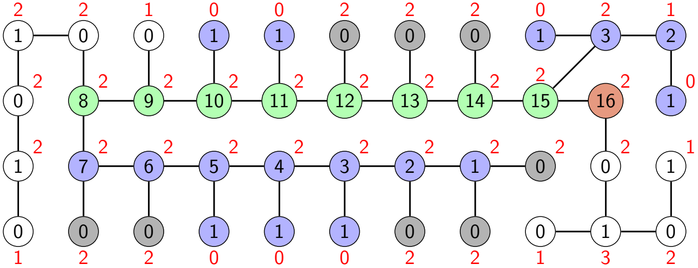
TSS-time in trees

- A vertex is *saturated* if the threshold is equal to the degree
- The maximum activation time is the size of a maximum path with non-saturated vertices plus 1
- Solvable in time $O(n)$



GTSS-time in trees

- A vertex is *saturated* if the threshold is equal to the degree
- Simulate the activation process from the forced vertices
- The maximum activation time is the maximum (among paths P of non-saturated and inactive vertices) of the size $|P|$ plus the beginning time of P plus 1.
- Solvable in time $O(n^2)$



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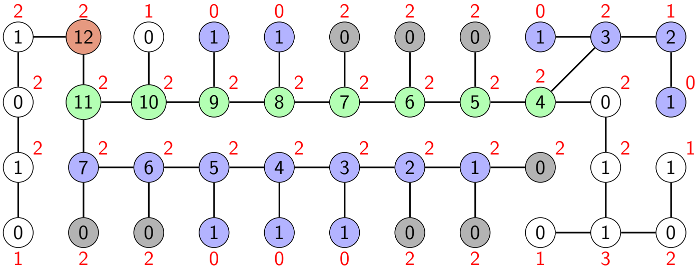

The END !!

THANK YOU !!

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