PSPACE-hardness of two coloring games

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Introduction

Proper coloring

- The vertices of graph are colored
- Two adjacent vertices must receive distinct colors
- $\chi(G)$: chromatic number (min number in a proper coloring)

Greedy coloring

- Proper vertex coloring / colors are integers
- Take an ordering of the vertices.
- A vertex must receive the minimum available color.
- $\Gamma(G)$: Grundy number (max number in a greedy coloring)

$$\chi(G) \leq \Gamma(G)$$

Two graph coloring games

- **Instance:** a graph G and a set C of colors/integers
- Two players **Alice** and **Bob** alternate their turns in choosing an uncolored vertex to be **proper colored** by an integer of C
- Alice **starts** and **she wins** if all vertices are successfully colored; Otherwise, Bob wins the game
- Zermelo-von Neumann Th.: Alice or Bob has a winning strat

Graph coloring game ($\chi_g(G) \geq \chi(G)$)

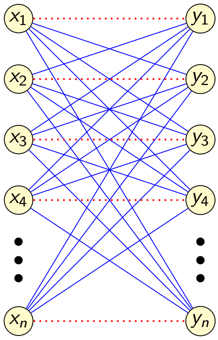
- Alice and Bob may use **any** possible integer of C
- **Game chromatic** number $\chi_g(G)$: minimum number of colors s.t. Alice has a winning strategy in the graph coloring game

Greedy coloring game ($\chi(G) \leq \Gamma_g(G) \leq \Gamma(G)$)

- Alice and Bob must use **the smallest** possible integer of C
- **Game Grundy number** $\Gamma_g(G)$: minimum number of colors s.t. Alice has a winning strategy in the greedy coloring game

An example for both coloring games

- Complete bipartite graph without a matching
- **If Alice is the first** to play, Bob can force n colors: just play in the non-neighbor of Alice's last vertex.
- **If Bob is the first** to play, Alice wins with 2 colors.
  Normal game: Alice colors non-neighbor with other color.
  Greedy game: color same side of Bob's first vertex.



Known results

Graph coloring game / Game chromatic number $\chi_g(G)$

- First considered by [Brams] and described by [Gardner'81, Math. Games column of Scientific American]
- Reinvented by [Bodlaender'91]: "*The complexity of the Color Construction Game is an interesting open problem*"
- forest ≤ 4 [Faigle...'93], outerplanar ≤ 7 [Kierstead...'94]
- $\chi_g \leq (\chi_a + 1)^2$ acyclic chromatic number χ_a [Dinski,Zhu'99]
- $\chi_g(P_k) \leq 3k + 2$ for partial k trees [Zhu'00]
- $\chi_g(G) \leq 5$ in cacti [Sidorowicz'07]
- Asympt. behavior $\chi_g(G(n,p))$ [Bohman, Frieze, Sudakov'08]
- Ex.value χ_g cartes prod K_2 w path/cycle/clique [Bartnick'08]
- Planar graphs: $\chi_g \leq 17$ [Zhu'08], $\chi_g \leq 13$ [Sekiguchi'14, girth≥ 4], $\chi_g \leq 5$ [Nakprasit'18, girth≥ 7]

- $\chi_g(F)$ poly forests no vertex deg 3 [Dunn et. al'15]: "*more than two decades later,* ***this question*** *remains open*".
- poly characterization game-perfect graphs [Andres,Lock'19]: "*the question of PSPACE-hardness remains open*".

Known results

Greedy coloring game / Game Grundy number $\Gamma_g(G)$

- Introduced by [Havet, Zhu'13]
- $\Gamma_g(G) = \chi(G)$ in cographs [Havet,Zhu'13]
- $\Gamma_g(F) \leq 3$ in forests [Havet, Zhu'13]
- $\chi_g(G) \leq 7$ in partial 2-trees [Havet, Zhu'13]
- Two questions of [Havet,Zhu'13]
  - (*) $\chi_g(G)$ is upper bounded by a function of $\Gamma_g(G)$?
  - (**) $\Gamma_g(G) \leq \chi_g(G)$ for every graph G?
- (*) $=$ NO [Krawczyk,Walczak'15]
- (**) is still open

Our results

Complexity results

- $\chi_g(G)$ is PSPACE-hard answer Bodlaender'91 open question
- $\Gamma_g(G)$ is PSPACE-hard
- Both decision problems are PSPACE-Complete

Exact/algorithmic results

- $\Gamma_g(G) = \chi(G)$ poly for split graphs
- $\Gamma_g(G) = \chi(G)$ poly for extended P_4-laden graphs, a class in the top of a hierarchy of graphs with few P_4's
- In both cases, Alice wins with $\chi(G)$ colors even if Bob can start the game and pass any turn

$\chi_g(G)$ is PSPACE-hard

Zhu'99 open question: Graph coloring game "*exhibits some strange properties*". Does Alice have a winning strategy with $k+1$ colors if she has one with k colors?

We define three decision problems for the graph coloring game:

- (Problem 1) Given G and k: $\chi_g(G) \leq k$?
- (Problem 2) Given G and k: Does Alice have a winning strategy with k colors?
- (Problem 3) Given G and $\chi(G)$: $\chi_g(G) = \chi(G)$?

Problems 1 and 2 are equivalent **iff** Zhu's question is true.
Problems 1 and 2 generalizations of Problem 3 - take $k = \chi(G)$

Problem 3 PSPACE-hard $\rightarrow$ Problems 1 and 2 PSPACE-hard

Reduce POSCNF $\rightarrow$ Problem 3: build G s.t. we know $\chi(G)$.

χ_g: Reduction from POS-CNF

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CNF formula, only positive variables, Alice and Bob alternate turns setting variables true or false. Alice wins if the formula is true.

Example

$(X_1 \vee X_2) \quad \wedge \quad (X_1 \vee X_3) \quad \wedge \quad (X_2 \vee X_4) \quad \wedge \quad (X_3 \vee X_4)$.

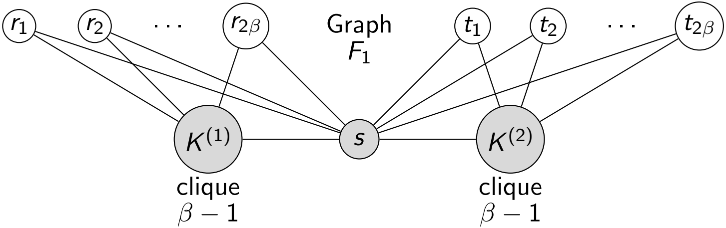
Bob has a winning strategy:

- X_1 True $\rightarrow X_4$ False;
- X_2 True $\rightarrow X_3$ False;
- X_4 True $\rightarrow X_1$ False;
- X_3 True $\rightarrow X_2$ False.

Good points

- POS-CNF is PSPACE-Complete
- If she/he has a winning strategy in POS-CNF, she/he also has a winning strategy if the opponent can pass turns.

χ_g: Important ingredient of the Reduction



Lemma:

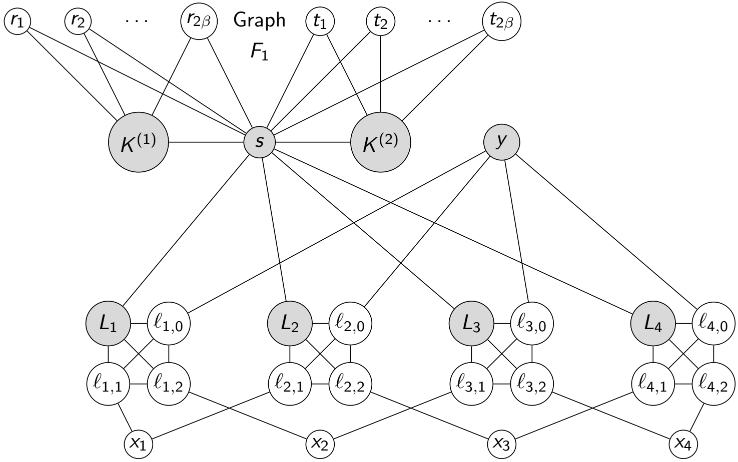
Alice has a winning strategy in F_1 with $2\beta - 1$ colors **iff** she colors vertex s first.

Proof:

If Alice does not color s first, Bob can color β vertices r_k/t_k, forcing 2β colors with clique $K^{(i)} \cup s$. If Alice colors s first, she can color $K^{(1)}$ and $K^{(2)}$ before Bob colors β vertices r_k/t_k.

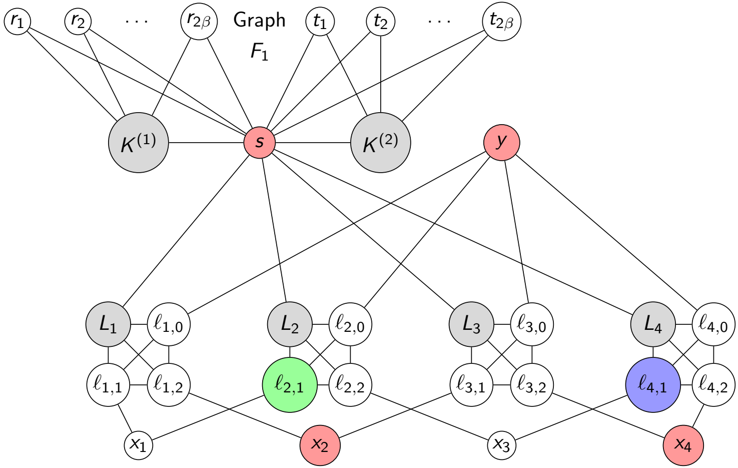
χ_g: Reduction from POS-CNF

$$(X_1 \vee X_2) \;\wedge\; (X_1 \vee X_3) \;\wedge\; (X_2 \vee X_4) \;\wedge\; (X_3 \vee X_4).$$



χ_g: Reduction from POS-CNF

$$(X_1 \vee X_2) \quad \wedge \quad (X_1 \vee X_3) \quad \wedge \quad (X_2 \vee X_4) \quad \wedge \quad (X_3 \vee X_4).$$



$\Gamma_g(G)$ is PSPACE-hard

Differently than the Graph Coloring Game, if Alice has a winning strategy with $k+1$ colors in the greedy coloring game she has one with k colors.

We define two decision problems for the greedy coloring game:

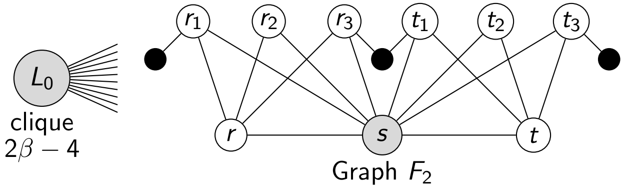
- (Problem 1') Given G and k: $\Gamma_g(G) \leq k$? That is: Does Alice have a winning strategy with k colors?
- (Problem 2') Given G and $\chi(G)$: $\Gamma_g(G) = \chi(G)$?

Problems 1' generalization of Problem 2' - take $k = \chi(G)$

Problem 2' PSPACE-hard $\rightarrow$ Problem 1' PSPACE-hard

Reduce POSCNF $\rightarrow$ Problem 2': build G s.t. we know $\chi(G)$.

Γ_g: Important ingredient of the Reduction



Lemma:

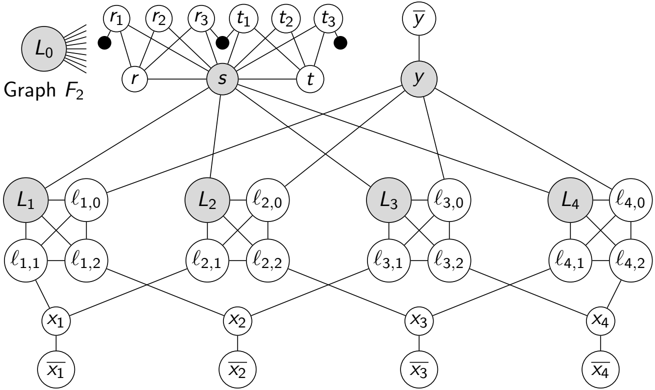
Alice has a winning strategy in F_2 with $2\beta - 1$ colors **iff** she colors vertex s first.

Proof:

If Alice does not color s first (assume r wlg), Bob colors t_2 and a black vertex with 1, forcing 4 colors in a triangle $s - t - t_i$. If Alice colors s first, she can color r and t with 2 or 3 (black vertices will be 1), forcing 3 colors.

Γ_g: Reduction from POS-CNF

$$(X_1 \vee X_2) \quad \wedge \quad (X_1 \vee X_3) \quad \wedge \quad (X_2 \vee X_4) \quad \wedge \quad (X_3 \vee X_4).$$



Positive results

[Havet,Zhu'13]: $\Gamma_g(G) = \chi(G)$ for cographs (no induced P_4)

Cographs: $\chi(G) = \Gamma(G)$. Then $\Gamma_g(G) = \chi(G)$ (even if Bob starts and can pass any turn).

Superclasses of cographs:

- P_4-sparse, P_4-laden, P_4-tidy
- $\Gamma(G)$ can be larger than $\chi(G)$ as much as desired

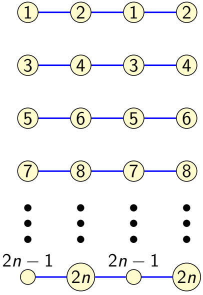
Split graphs:

- partition $V = C \cup S$: Clique C and indepedent set S
- $\chi(G) \;\leq\; \Gamma(G) \;\leq\; \chi(G) + 1$

We prove $\Gamma_g(G) = \chi(G)$ (even if Bob starts / can pass any turn) in split graphs and extended P_4-laden graphs.

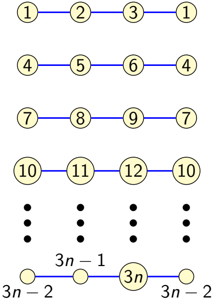
P_4-sparse example: join of n P_4's

P_4-sparse graph G



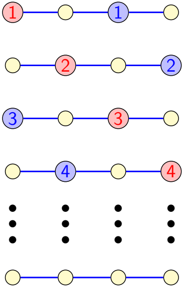
$\chi(G) = 2n$

P_4-sparse graph G



$\Gamma(G) = 3n$

P_4-sparse graph G



$\Gamma_g(G) = 2n$

Graph classes with few P_4's

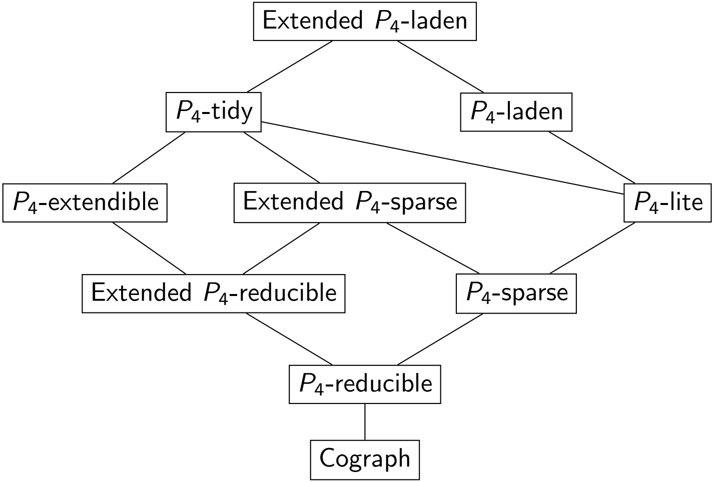


Figure: Hierarchy of graphs with few P_4's.

Extended P_4-laden graphs

Decomposition theorem [Giakoumakis'96]

G is extended P_4-laden **iff** one of the following holds:

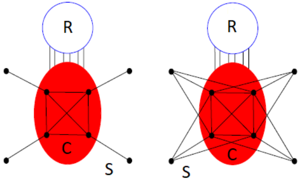
(a) G is the disjoint union or the join of two non-empty extended P_4-laden graphs;

(b) G is a quasi-spider or a pseudo-split graph (R, C, S) such that $G[R]$ is an extended P_4-laden graph;

(c) G is isomorphic to C_5, P_5, $\overline{P_5}$, or has at most one vertex.

Operations: union, join, spider

- **Union** $G = G_1 \cup G_2$: No edge between G_1 and G_2.
- **Join** $G = G_1 \vee G_2$: All edges between G_1 and G_2.



G is a **pseudo-split** (R, C, S) if:

- C induces a clique and S induces an independent set
- All edges from R to C and no edges from R to S

G is a **spider** if it is a pseudo-split (R, C, S) st:

- $C = \{c_1, \dots, c_k\}$ and $S = \{s_1, \dots, s_k\}$ for some $k \geq 2$
- **Thin spider**: s_i is adjacent to c_j if and only if $i = j$
- **Thick spider**: s_i is adjacent to c_j if and only if $i \neq j$

$\Gamma'_g(G)$: Bob can start and pass any turn

Let $\Gamma'_g(G)$ be the minimum number of colors st Alice has a winning strategy in the greedy coloring game even if Bob can start and pass any turn. $\chi(G) \leq \Gamma_g(G) \leq \Gamma'_g(G) \leq \Gamma(G)$.

Union and Join

- $\Gamma'_g(G_1 \cup G_2) \leq \max\{\Gamma'_g(G_1), \Gamma'_g(G_2)\}$;
- $\Gamma'_g(G_1 \vee G_2) \leq \Gamma'_g(G_1) + \Gamma'_g(G_2)$.

Pseudo-split, quasi-spider, C_5, P_5, $\overline{P_5}$

- G pseudo-split $(R, C, S) \implies \Gamma'_g(G) \leq \Gamma'(G[R]) + |C|$
- G quasi-spider or $G \in \{C_5, P_5, \overline{P_5}\} \implies \Gamma'_g(G) = \chi(G)$

Extended P_4-laden

Applying the decomposition, we prove by induction that $\Gamma'_g(G) = \chi(G)$ for any extended P_4-laden G.

Conclusion

Complexity results

- Game chromatic number $\chi_g(G)$ is PSPACE-hard, answering Bodlaender's 1991 open question
- Game Grundy number $\Gamma_g(G)$ is PSPACE-hard
- The Graph Coloring Game and the Greedy Coloring Game are PSPACE-Complete

Exact/algorithmic results for Γ_g

- $\Gamma_g(G) = \chi(G)$ poly for split graphs
- $\Gamma_g(G) = \chi(G)$ poly for extended P_4-laden graphs, a class in the top of a hierarchy of graphs with few P_4's
- In both cases, Alice wins with $\chi(G)$ colors even if Bob can start the game and pass any turn

THANK YOU !!