A note on random k-dimensional posets

Rudini Sampaio (UFC, Fortaleza, Brazil)

This is a joint work with
Ricardo Corrêa (UFC, Fortaleza, Brazil)
Carlos Hoppen (UFRGS, Porto Alegre, Brazil)
Yoshiharu Kohayakawa (USP, São Paulo, Brazil)

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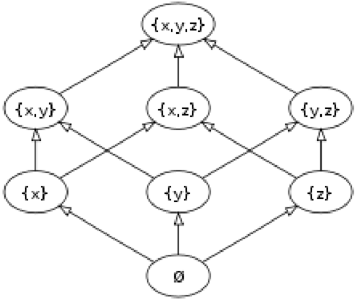
Basic definitions

Permutation σ on $[n] = \{1, 2, \ldots, n\}$.

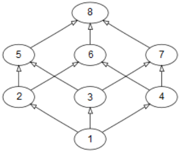
$$(4, 5, 2, 3, 6, 1) \text{ is a permutation on } [6].$$

Partially ordered sets (or just **posets**):
reflexive, antisymmetric and transitive binary relation.



Permutations versus Posets

Posets on $[n]$



Realizer: set of permutations the intersection of which generates the poset.

$$(1, 2, 3, 5, 4, 6, 7, 8)$$

$$(1, 4, 3, 7, 2, 6, 5, 8)$$

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The dimension of a poset is the minimum size of a realizer.

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Question: How can we estimate some parameter?

Question: How can we test if it satisfies some property?

Question: How can we obtain some optimized substructure?

Convergent sequences

Given a sequence of objects (graphs, permutations, posets),

Question: when does it converge?

Question: There exists some limit object?

Limit permutation

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Z-random permutation $\sigma(n, Z)$: Generate according to Z n pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$. Then, $\sigma(n, Z) = S \circ R^{-1}$, where R and S are the ranking of $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$, respectively.

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Example:

$$\begin{pmatrix} X & 0.62 & 0.44 & 0.15 & 0.87 & 0.53 \\ Y & 0.33 & 0.11 & 0.98 & 0.25 & 0.67 \end{pmatrix} \to \begin{pmatrix} R & 4 & 2 & 1 & 5 & 3 \\ S & 3 & 1 & 5 & 2 & 4 \end{pmatrix}$$

$\sigma(n, Z) = (5, 1, 4, 3, 2)$

Subpermutations

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$t(\tau, \sigma)$ is the proportion of subpermutations τ in σ.

Definition

A sequence of permutations (σ_n) is said to converge if, for every fixed permutation τ, the real sequence $(t(\tau, \sigma_n))_{n\in\mathbb{N}}$ converges.

Convergent permutation sequences

Theorem (Hoppen et al., 2010)

For every convergent permutation sequence (σ_n), *there exists a limit permutation* $Z = (X, Y)$, *such that, for every permutation* τ,

$$\lim_{n \to \infty} t(\tau, \sigma_n) \; = \; t(\tau, Z) \; := \; \mathbb{P}\Big(\sigma(k, Z) = \tau\Big),$$

where k *is the size of* τ.

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where k is the size of τ.

Theorem (Hoppen et al., 2010)

Let Z be a limit permutation. The sequence $(\sigma(n, Z))_{n=1}^{\infty}$ converges to Z with probability one.

Limits for k-dimensional posets

A sequence of k-dimensional posets can be represented by k sequences of permutations.

This suggests a limit for k-dimensional poset sequences.

Question: When does such a sequence converge?

Question: What kind of limit we have?

Limit k-dimensional poset (or k-kernel)

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$P(n, Z)$ is the poset $([n], \prec_P)$ such that $i \prec_P j$ if and only if $Y^{(i)} < Y^{(j)}$ (if and only if every coordinate of $Y^{(i)}$ is smaller than the corresponding coordinate of $Y^{(j)}$).

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This model generalizes the random k-dimensional poset model (just take the k-kernel Z_{ind} where $X_1, \ldots, X_k$ are independent).

Convergent k-dim. poset sequences

Definition

A sequence of k-dimensional posets (B_n) is said to converge if, for every fixed poset P, the real sequence $(t(P, B_n))_{n\in\mathbb{N}}$ converges.

$t(P, B_n)$ is the probability that a random subposet of B_n has the same order of P.

Convergent k-dim. poset sequences

Definition

A sequence of k-dimensional posets (B_n) is said to converge if, for every fixed poset F, the real sequence $(t(F, B_n))_{n\in\mathbb{N}}$ converges.

Theorem (main result)

For every convergent k-dimensional poset sequence (B_n), there exists a k-kernel $Z = (X_1, \ldots, X_k)$, such that, for every poset F,

$$\lim_{n\to\infty} t(F, B_n) \ = \ t(F, Z) \ := \ \mathbb{P}\Big(P(m, Z) = F\Big),$$

where m is the size of F.

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where k is the size of F.

Uniqueness of the limit

Definition

Let $Y(Z)$ be a random point in $[0,1]^k$ generated according to the k-kernel Z. The rectangular distance between k-kernels Z and Z':

$$d_{\square}(Z, Z') = \sup_{\Delta \in I[0,1]^k} \Big| \mathbb{P}\big(Y(Z) \in \Delta\big) - \mathbb{P}\big(Y(Z') \in \Delta\big) \Big|,$$

where $I[0,1]^k$ is the set of all k-dimensional intervals of $[0,1]^k$.

Uniqueness of the limit

Definition

Let $Y(Z)$ be a random point in $[0,1]^k$ generated according to the k-kernel Z. The rectangular distance between k-kernels Z and Z':

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where $I[0,1]^k$ is the set of all k-dimensional intervals of $[0,1]^k$.

Theorem (main result)

$\delta_\square(Z, Z') = 0$ if and only if $t(P, Z) = t(P, Z')$ for every poset P.

That is, if $(B_n) \to Z$ and $(B_n) \to Z'$, then $\delta_\square(Z, Z') = 0$.

Quasirandomness

Definition

A sequence (B_n) of k-dimensional posets is quasirandom if it converges to the k-kernel Z_{ind}.

Theorem

There exists a sequence (R_n)*, where* $R_n = (\sigma_{n,1}, \ldots, \sigma_{n,k})$ *is a realizer of* B_n*, such that, for every pair* $i \neq j \in [k]$*, the permutation sequence* $(\sigma_{n,i} \circ \sigma_{n,j}^{-1})$ *is quasirandom in the sense of Cooper.*

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Parameter testing through subposets

Objective: accurately predict the value of a parameter $f(B)$ by looking at a randomly chosen subposet of much smaller size.

Definition

A parameter f is k-dim. testable if,
For every $\epsilon > 0$,
There exist positive integers $t < n_0$ s.t.:

If B is a k-dimensional poset of length $n > n_0$, then

$$\mathbb{P}\Big(|f(B) - f(sub(t, B))| > \varepsilon\Big) \leq \varepsilon.$$

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A parameter f is testable if it is k-dim testable, for every k.

Characterization of testable parameters

Theorem

A bounded poset parameter is k-dim. testable if and only if the sequence $(f(B_n))_{n\in\mathbb{N}}$ converges for every convergent sequence $(B_n)_{n\in\mathbb{N}}$ of k-dimensional posets.

A poset parameter f is bounded if there is a constant M such that $|f(B)| < M$ for every poset σ.

Immediate consequences

Testable Poset Parameters

- The subposet density $f_P(B) = t(P, B)$ for any fixed P.
- The density of pairs.

NOT Testable Poset Parameters (through subposets)

- The height (over n).
- The width (over n).

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