Graphs with few P_4's under the convexity of paths of order three

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May 30th, 2012

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- $C \in \mathcal{C}$ is called *convex*

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- Notation: $H(S)$
- Also, $H(S)$ is the intersection of all convex sets containing S

Interval Function

A graph convexity $(G, \mathcal{C})$ might be related to a function I, called *Interval Function*

$$I : 2^V \rightarrow 2^V$$

where

1. if $S \in \mathcal{C}$, then $I(S) = S$
2. $S \subset I(S) \subseteq H(S)$ otherwise

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$H(S)$ can be obtained from S by iteratively applying the interval function until a convex set is reached.

Interval Sets and Hull Sets

Graph convexity $(G, \mathcal{C})$ and $S \subseteq V(G)$

1. S is an *interval set* if $I(S) = V(G)$
2. S is a *hull set* if $H(S) = V(G)$

Convexity Parameters

Interval number cardinality of minimum interval set

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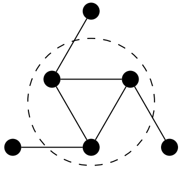
Radon number smallest k such that if $S \subseteq V(G)$ and $|S| \geq k$ then S can be partitioned into S_1 and S_2 such that $H(S_1) \cap H(S_2) \neq \emptyset$

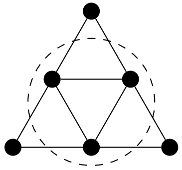
P_3 Convexity

The interval function assigns , for each $S \subseteq V(G)$, all vertices adjacent to two distinct vertices of S.

CONVEX



NOT CONVEX



Basic problems

Given $S \subseteq V(G)$:

- Compute $I(S)$
- Decide if S is convex
- Decide if S is an interval set
- Compute $H(S)$
- Decide if S is a hull set

All polynomial.

Hull number - Motivation

Spread of infection in a square grid

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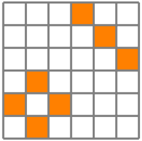
B. Bollobás
Coffee time in Memphis.
2006.

Infection process



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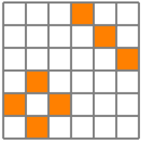

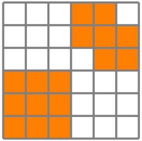

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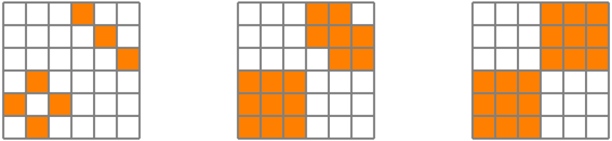

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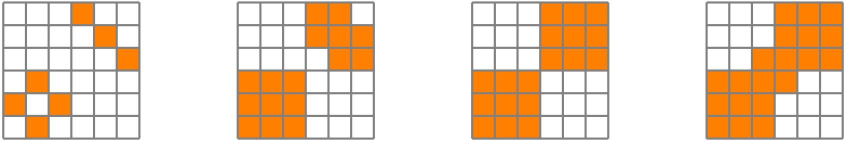

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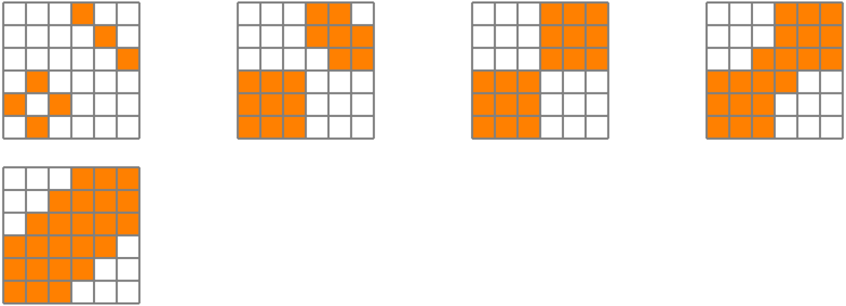

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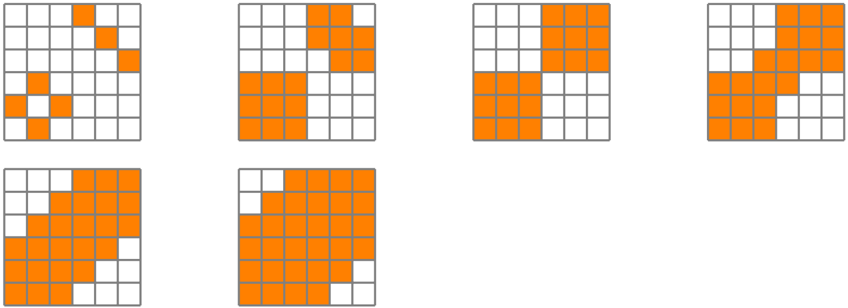

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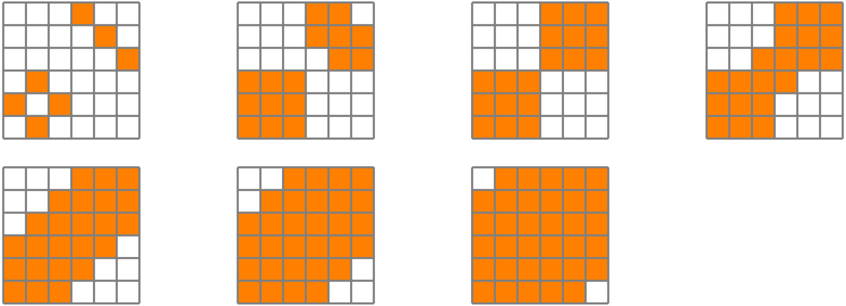

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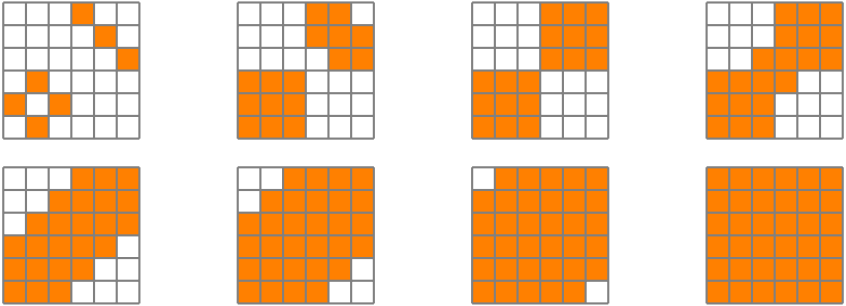

Convexity parameters - General graphs

Theorem

It is NP-hard to determine the **hull number**, **interval number**, **convexity number**, **Carathéodory number** or **Radon number** of a general graph.

- Barbosa, Coelho, Dourado, Rautenbach, Szwarcfiter, *On the Caratheodory Number for the Convexity of Paths of Order Three*, to appear.
- Barbosa, Coelho, Dourado, Rautenbach, Szwarcfiter, Toman, *On the Radon Number for the Convexity of Paths of Order Three*, LATIN 2012.
- Centeno, Dantas, Dourado, Rautenbach, Szwarcfiter, *Convex Partitions of Graphs Induced by Paths of Order Three*, DMTCS 2010.
- Centeno, Dourado, Penso, Rautenbach, Szwarcfiter, *Irreversible Conversion of Graphs*, TCS 2011.

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It is polynomial to determine the **hull number**, **interval number**, **convexity number**, **Carathéodory number** or **Radon number** for cographs.

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Our results

Theorem

There is a $O(n)$ time algorithm to determine the **hull number**, **interval number**, **convexity number**, **Carathéodory number** or **Radon number** for $(q, q-4)$-graphs.

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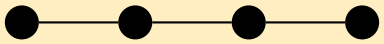
Theorem

The Carathéodory number is at most 3 for every cograph, P_4-sparse graph and every connected $(q, q-4)$-graph with at least q vertices.

$(q, q-4)$-graphs generalize cographs

P_4



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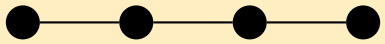

P_4-free graphs (cographs)

No induced P_4s.

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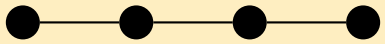

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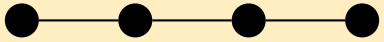

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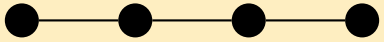

P_4-free graphs (cographs) $= (4, 0)$ graphs

No induced P_4s.

P_4-sparse graphs $= (5, 1)$ graphs

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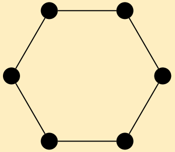
$(q, q-4)$ graphs

Set of $\leq q$ vertices induces $\leq q-4$ P_4s.

$(q, q-4)$-graphs: P_4-connectivity

- G is p-connected if, for any partition of $V(G)$ into non-empty A and B, there exists at least one P_4 with vertices in both A and B.

p-connected



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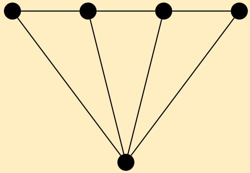
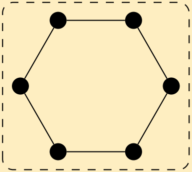


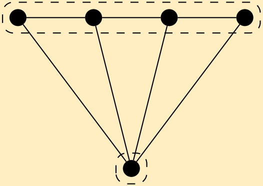
$(q, q-4)$-graphs: P_4-connectivity

- G is p-connected if, for any partition of $V(G)$ into non-empty A and B, there exists at least one P_4 with vertices in both A and B.
- A p-component is a maximal p-connected subgraph.

p-connected: one p-component



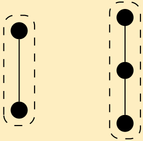
not p-connected: two p-component



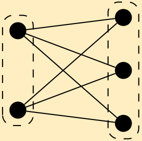
$(q, q-4)$-graphs: Primeval decomposition

For any graph G, exactly one of the following occurs:

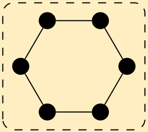
Disconnected



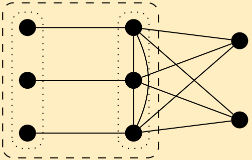
Codisconnected



p-connected



Separable p-component



B. Jamison and S. Olariu
P-components and the homogeneous decomposition of graphs.
SIAM Journal on Discrete Math, 1995.

Application of decomposition to hull number

Let $h(G)$ be the hull number of G.

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If $G = G_1 \vee G_2$ and

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then $h(G) = 2$.

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If $G = G_1 \vee G_2$ and

- $|V(G_1)| = 1$ and G_2 has k components

then $h(G) = max\{2, k\}$.

To wrap things up

Theorem:

If G is $(q, q-4)$, p-connected and has $\geq q$ vertices, then G is isomorphic to a spider graph.

L. Babel e S. Olariu
On the structure of graphs with few P_4s.
Discrete Applied Math, 1998.

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- In leaf nodes of the decomposition (G is p-connected), we have a formula for $h(G)$ when G is a spider graph. We find it by brute force if G has less than q vertices.

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- We can find $h(G)$ when G has a separable p-component.
- We complete the algorithm using the primeval decomposition and dynammic programming.

Future work

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  - What about planar bipartite graphs?
  - OBS: Square grids are planar bipartite.

The End

Thank You!!