Limits of permutation and k-dimensional poset sequences

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This includes joint works with
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TU Chemnitz, September 06, 2018

Brazilian cities

Figure: Fortaleza - Porto Alegre: 3200 km

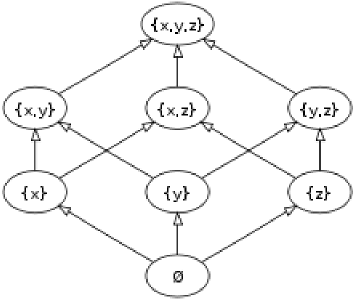
Basic definitions

Graph G with n vertices: $V(G) = [n] = \{1, 2, \dots, n\}$.

A permutation σ on $[n]$ is a bijective function of $[n]$ into $[n]$.

$(4, 5, 2, 3, 6, 1)$ is a permutation on $[6]$.

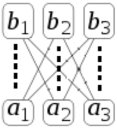
Partially ordered set (or simply **poset**):
reflexive, antisymmetric and transitive binary relation.



Basic definitions

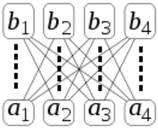
Realizer of a poset: set of complete orders the intersection of which generates the poset.

The dimension of a poset is the minimum size of a realizer.
Examples:



$$a_2 < a_3 < b_1 < a_1 < b_2 < b_3$$
$$a_1 < a_3 < b_2 < a_2 < b_1 < b_3$$
$$a_1 < a_2 < b_3 < a_3 < b_1 < b_2$$

Dimension 3



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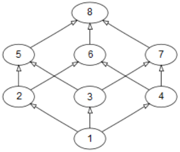
Dimension 4

Basic definitions

Posets on $[n] = \{1, 2, \ldots, n\}$



Realizer of a poset: set of permutations the intersection of which generates the poset.

$$(1, 2, 3, 5, 4, 6, 7, 8)$$

$$(1, 4, 3, 7, 2, 6, 5, 8)$$

$$(1, 2, 4, 6, 3, 5, 7, 8)$$

The dimension of a poset is the minimum size of a realizer.

Large Graphs, Permutations, Posets,...

Given a large discrete structure (graph, permutation, poset, . . .),

Question: How can we estimate some parameter?

Question: How can we test if it satisfies some property?

Question: How can we obtain some optimized substructure?

Graph testability

Some results

- Every monotone graph property is testable (Alon-Shapira'08)
- Every hereditary graph property is testable (Alon-Shapira'09).
- Every hereditary hypergraph property is testable (Rödl-Schacht'09, Austin-Tao'10).

Testable graph property $\mathcal{P}$

For any $\varepsilon > 0$, there are $q_{\mathcal{P}}(\varepsilon) > 0$ (query complexity) and a randomized algorithm $\mathcal{T}_{\mathcal{P}}$ (tester), s.t. for any graph G:

1. $\mathcal{T}_{\mathcal{P}}$ may perform at most $q_{\mathcal{P}}$ queries in the input graph G;
2. If G satisfies $\mathcal{P}$, then $\mathcal{T}_{\mathcal{P}}$ answers YES with prob. $2/3$.
3. If G is ε-far from $\mathcal{P}$, then $\mathcal{T}_{\mathcal{P}}$ answers NO with prob. $2/3$.

- G is ε-far from satisfying $\mathcal{P}$ if $d_1(G, \mathcal{P}) \geq \varepsilon$
- $d_1(G, \mathcal{P}) = \min\{d_1(G, G') : V(G) = V(G') \text{ and } G' \in \mathcal{P}\}$
- $d_1(G, G') = |E(G) \triangle E(G')| / \binom{n}{2}$, where $V(G) = V(G') = [n]$

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Estimable graph parameter $\rho(G)$

For any $\varepsilon > 0$, $\exists$ n_0 and q_ρ, s.t. $\rho(G)$ can be approx. up to an additive error ε with prob. $2/3$ by randomized algorithm that only has access to q_ρ vertices of G chosen uniformly at random.

Convergent sequences

Testable properties, estimable parameters, . . .

Some results are related to "convergent sequences".

Given a sequence of objects (graphs, permutations, posets),

Question: when does it converge?

Question: There exists some limit object?

Convergent sequence of graphs

Definition

A graph sequence (G_n) is convergent if, for every simple graph F, the sequence of densities $t(F, G_n)$ converges.

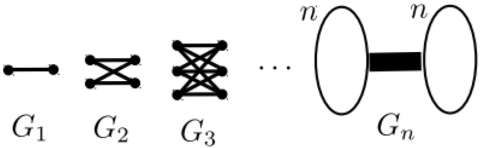
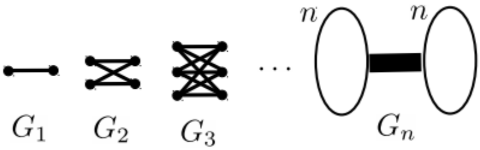
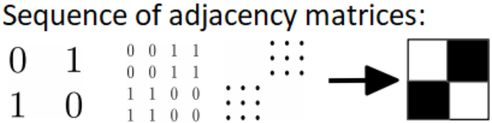


Figure: Sequence of complete bipartite graphs

- Increasing $|V(G_n)| \to \infty$
- $t(F, G_n)$: density of F-copies in $G_n = \mathbb{P}(G[random\ set] \cong F)$

Limit object - *graphons*





Limit object - *graphons*

Limit object - *Graphon*

A *Graphon* W is a symmetric measurable function from $W : [0,1]^2 \to [0,1]$.

W-random graph $G(n, W)$

We generate uniformly random $X_1, X_2, ..., X_n$ in $[0,1]$. The graph $G(n, W)$ has vertex set $[n] = \{1, \ldots, n\}$ and ij is an edge with prob. $W(X_i, X_j)$.

Limit object - *graphons*

Existence [Lovász, Szegedy, 2008]

If (G_n) is a convergent graph sequence, then there exists a *graphon* W such that, for every simple graph F with k vertices,

$$\lim_{n\to\infty} t(F, G_n) = t(F, W) := \mathbb{P}(G(k, W) \cong F)$$

Every graphon is a limit [Lovász, Szegedy, 2008]

With prob. 1,

$$(G(n, W)) \rightarrow W$$

Limit object - *graphons*

The limit is almost unique [Borgs et al., 2010]

If W_1 and W_2 are limits of some convergent graph sequence, then $\delta_\Box(W_1, W_2) = 0$.

Cauchy sequence [Borgs et al., 2010]

(G_n) is convergent if and only if it is Cauchy with respect to the distance $\delta_\Box$

$$d_\Box(W, W') = \sup_{A,B\subseteq[0,1]} \left| \int_A \int_B \Big(W(x,y) - W'(x,y) \Big) dx\, dy \right|,$$

[Alon, Shapira, 2009] and [Lovász, Szegedy, 2011]

Every hereditary graph property is testable.

Permutations

- The permutation τ on $[m]$ is a subpermutation of σ on $[n]$ if there is an m-tuple $x_1 < \cdots < x_m \in [n]^m$ such that $\tau(i) < \tau(j)$ if and only if $\sigma(x_i) < \sigma(x_j)$ for every $(i,j) \in [m]^2$.

Example: $\tau = (3,1,4,2)$, $\sigma = (5,6,2,4,7,1,3)$.

$$\sigma = (5,6,2,4,7,1,3).$$

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Permutations

Let $\Lambda(\tau, \sigma)$ be the number of occurrences of the permutation τ on $[k]$ in the permutation σ on $[n]$. The density of the permutation τ as a subpermutation of σ is given by

$$t(\tau, \sigma) = \begin{cases} \binom{n}{k}^{-1} \Lambda(\tau, \sigma) & \text{if } k \leq n \\ 0 & \text{if } k > n. \end{cases}$$

If τ is a fixed permutation and $(\sigma_n)_{n \in \mathbb{N}}$ is a convergent sequence, it would be natural to require that the real sequence $(t(\tau, \sigma_n))_{n \in \mathbb{N}}$ converges.

Convergent sequence of permutations

Definition

A sequence of permutations (σ_n) is said to converge if, for every fixed permutation τ, the real sequence $(t(\tau, \sigma_n))_{n \in \mathbb{N}}$ converges.

Example: Let $\sigma_n = (1, 2, \ldots, n)$ for every positive integer n.

For a permutation τ,

$$t(\tau, \sigma_n) = \begin{cases} 1, & \text{if } \tau = (1, \ldots, k),\ k \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

$(\sigma_n)_{n \in \mathbb{N}}$ is convergent!

Convergent sequence of permutations

Example: For every integer n, let π_n be a permutation on $[n]$ chosen uniformly at random.

For a permutation τ,

$$\mathbb{E}(t(\tau, \pi_n)) = \begin{cases} 1/m!, & \text{if } m \leq n; \\ 0, & \text{if } m > n. \end{cases}$$

With concentration arguments, we may show that $(\pi_n)_{n\in\mathbb{N}}$ converges with probability 1.

A limit for a permutation sequence?

Question: Is there a limit for a convergent permutation sequence?

C. Hoppen and Y. Kohayakawa and C. G. Moreira and R. M. Sampaio
Limits of permutation sequences
Journal of Combinatorial Theory series B, 2013

Limit object: *permutons*

A permuton is a pair $Z = (X, Y)$ of uniformly random variables X and Y in $[0, 1]$. That is, Z is a measure in $[0, 1]^2$ with uniform marginals, given by the joint distribution function of (X, Y).

Example

$$\sigma = \left\{ \binom{1}{1}, \binom{2}{4}, \binom{3}{3}, \binom{4}{5}, \binom{5}{2} \right\}$$

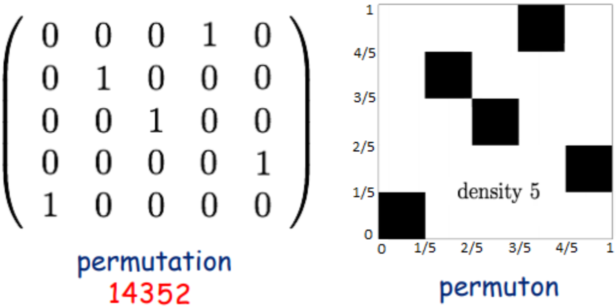

Figure: Permutons: limit objects of permutation sequences

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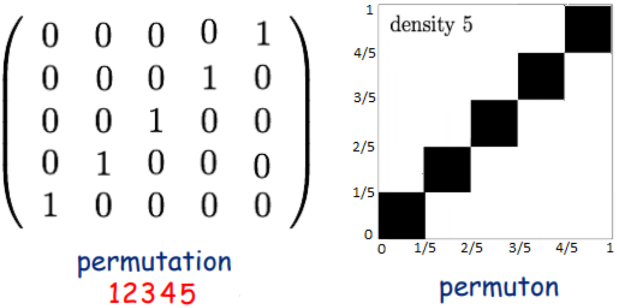

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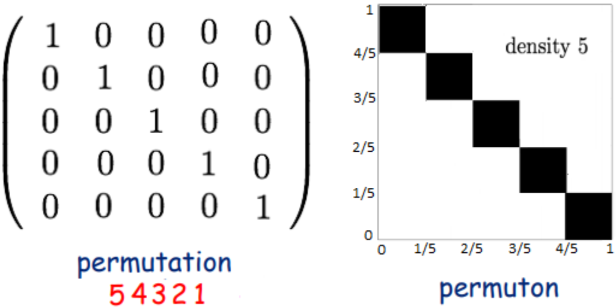
Example

$\sigma = \left\{ \binom{1}{5}, \binom{2}{4}, \binom{3}{3}, \binom{4}{2}, \binom{5}{1} \right\}$



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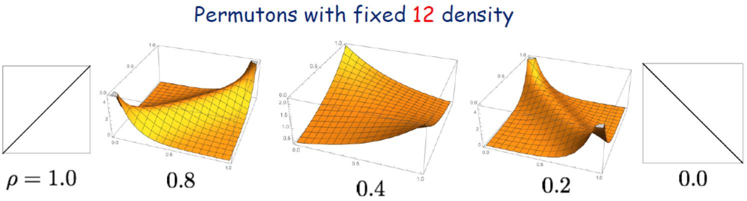


Images from a talk of Peter Winkler at Permutation Patterns (Reykjavik, 2017)

Figure: Permutons: limit objects of permutation sequences

Limit object: *permutons*

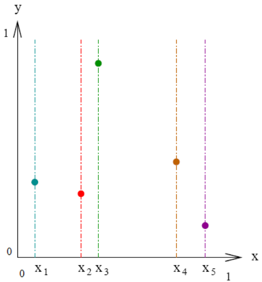
Z-random Permutation $\sigma(n, Z)$

We generate n pairs according to the distribution of (X, Y): $(x_1, y_1), \dots, (x_n, y_n)$. We sort these pairs in increasing order of the first coordinate: $(x'_1, y'_1), \dots, (x'_n, y'_n)$. The permutation $\sigma(n, Z)$ is the relative order defined by the second coordinates $(y'_1, \dots, y'_n)$.

Example:

$$\begin{pmatrix} X & 0.15 & 0.44 & 0.53 & 0.62 & 0.87 \\ Y & 0.33 & 0.25 & 0.98 & 0.67 & 0.11 \end{pmatrix} \to \begin{pmatrix} R & 1 & 2 & 3 & 4 & 5 \\ S & 3 & 2 & 5 & 4 & 1 \end{pmatrix}$$

$$\sigma(n, Z) = (3, 2, 5, 4, 1)$$



(3,2,5,4,1)

Limit object: *permutons*

Existence [Hoppen et al., 2013]

If (σ_n) is a convergent sequence of permutations, then there exists a limit permutation Z such that, for evey permutation τ, we have

$$\lim_{n\to\infty} t(\tau, \sigma_n) = t(\tau, Z) := \mathbb{P}(\sigma(k, Z) = \tau)$$

The limit is unique [Hoppen et al., 2013]

If a permutation sequence converges to permutons Z and Z', then Z and Z' differ in at most a set of Lebesgue measure zero (considering Z the joint distribution of (X, Y)).

Every Z is a limit [Hoppen et al., 2013]

If Z is a limit permutation, then $(\sigma(n, Z)) \to Z$

Limit object: *permutons*

Rectangular distance ($d_\square$)

Given permutations σ and π on $[n]$:

$$d_\square(\sigma,\pi) = \frac{1}{n} \max_{S,T \in I[n]} \Big| |\sigma(S) \cap T| - |\pi(S) \cap T| \Big|$$

Cauchy sequence [Hoppen et al., 2013]

A permutation sequence (σ_n) is convergent if and only if it is Cauchy with respect to $d_\square$.

Limit object: *permutons*

Testability in the rectangular distance

Every hereditary permutation property is weakly testable (according to the rectangular distance).

C. Hoppen, Y. Kohayakawa, C. G. Moreira and R. Sampaio
Testing permutation properties through subpermutations
Theoretical Computer Science, 2011 (SODA-2010)

Testability in the edit distance

Every hereditary permutation property is strongly testable (according to the Kendall tau distance).

T. Klimosová and D. Král'
Hereditary properties of permutations are strongly testable
SODA, 2014.

Permutation parameters

Example: $fp(\sigma)$ is the number of fixed points of σ.

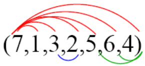
$$\sigma = (7, 1, 3, 2, 5, 6, 4) \qquad fp(\sigma) = 3$$

Example: $ord(\sigma)$ is the largest increasing subpermutation of σ.

$$\sigma = (7, 1, 3, 2, 5, 6, 4) \qquad ord(\sigma) = 4$$

Example: $inv(\sigma)$ is the number of inversions in σ.

$$\sigma = (7, 1, 3, 2, 5, 6, 4) \qquad inv(\sigma) = 9$$



Parameter testing

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Question: Can one accurately predict the value of a parameter $f(\sigma)$ in constant time for every permutation σ?

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Parameter Testing through subpermutations

Question: Can one accurately predict the value of a parameter $f(\sigma)$ by looking at a randomly chosen subpermutation of constant size?

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$sub(k, \sigma)$: random subpermutation of σ on $[k]$ (uniformly chosen)

$$\sigma = (5, 7, 2, 10, 1, 4, 8, 6, 3, 9) \qquad sub(4, \sigma) = (2, 4, 1, 3)$$

Parameter testing through subpermutations

Objective: accurately predict the value of a parameter $f(\sigma)$ by looking at a randomly chosen subpermutation of much smaller size.

Definition

A parameter f is testable if,
For every $\epsilon > 0$,
There exist positive integers $k < n_0$ s.t.:

If σ is a permutation of length $n > n_0$, then

$$\mathbb{P}\Big(|f(\sigma) - f(sub(k,\sigma))| > \varepsilon\Big) \leq \varepsilon.$$

Characterization of testable parameters

Theorem

A bounded permutation parameter is testable if and only if the sequence $(f(\sigma_n))_{n\in\mathbb{N}}$ converges for every convergent sequence $(\sigma_n)_{n\in\mathbb{N}}$ of permutations.

A permutation parameter f is bounded if there is a constant M such that $|f(\sigma)| < M$ for every permutation σ.

Immediate consequences

Testable Permutation Parameters

- The subpermutation density $f_\tau(\sigma) = t(\tau, \sigma)$ for any fixed τ.
- The inversion density $inv(\sigma) = t((2,1), \sigma)$.

NOT Testable Permutation Parameters (through subpermutations)

- The fixed-point density.
- The density of a longest increasing subsequence.

Property testing through subpermutations

We now want to look at more general properties of a permutation:

- Does it satisfy a given condition?
- Does it contain or avoid a given set of patterns?

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Property testing through subpermutations

We now want to look at more general properties of a permutation:

- Does it satisfy a given condition?
- Does it contain or avoid a given set of patterns?

Question: Can one predict the answer of such a question accurately by looking at a small subpermutation?

Modified question: Can one at least predict accurately if a permutation σ satisfies a property $\mathcal{P}$ or is far from satisfying it by looking at a small subpermutation?

Property testing through subpermutations

More precisely: a permutation property $\mathcal{P}$ is testable if, for every $\epsilon > 0$, there exist $k \leq n_0$ s.t. if σ is a permutation on $[n]$ with $n \geq n_0$, then with probability $\geq 1 - \epsilon$:

(i) σ satisfies $\mathcal{P}$ $\implies$ $sub(k, \sigma)$ satisfies $\mathcal{P}$

(ii) σ is ϵ-far from satisfying $\mathcal{P}$ $\implies$ $sub(k, \sigma)$ does not satisfy $\mathcal{P}$

σ is ϵ-far from satisfying $\mathcal{P}$ if (weak testable)

$$d_{\square}(\sigma, \mathcal{P}) = \min\{d_{\square}(\sigma, \pi) \ : \ \pi \text{ on } [n] \text{ satisfies } \mathcal{P}\} \geq \epsilon.$$

σ is ϵ-far from satisfying $\mathcal{P}$ if (strong testable)

$$d_1(\sigma, \mathcal{P}) = \min\{d_1(\sigma, \pi) \ : \ \pi \text{ on } [n] \text{ satisfies } \mathcal{P}\} \geq \epsilon.$$

Hereditary permutation properties

A permutation property $\mathcal{P}$ is hereditary if, whenever σ satisfies $\mathcal{P}$, then all its subpermutations satisfy $\mathcal{P}$.

Example: The property of avoiding a fixed pattern is hereditary.

Theorem (Hoppen et al., 2013)

Every hereditary permutation property is weakly testable.

Hereditary permutation properties

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Theorem (Klimosová and Král, SODA-2014)

Every hereditary permutation property is strongly testable.

Permutation quasirandomness

We say that a permutation sequence (σ_n) is quasirandom if $t(\pi, \sigma_n) \to 1/|\pi|!$. A classic question is whether we can determine if a sequence is quasirandom only checking few subpermutations.

Conjecture (Graham, 2004)

If $t(\pi, \sigma_n) \to 1/4!$ for every permutation π of size 4, then σ_n is quasirandom.

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Conjecture (Graham, 2004)

If $t(\pi, \sigma_n) \to 1/4!$ *for every permutation* π *of size 4, then* σ_n *is quasirandom.*

Theorem (Král and Pikhurko, 2013)

Graham's conjecture is true.

Posets

Janson [Combinatorica, 2011] focused in general poset sequences and obtained similar results:

- Defined convergence for poset sequences, based on density of subposets
- Defined a limit object: a measurable function defined on the Cartesian product of an ordered probability space with some additional properties: non-intuitive for a combinatorialist
- Proved that any convergent poset sequence has a limit object
- Conjectured that the ground set of the ordered probability spaces can be always $[0,1]$ with Lebesgue measure.

Hladky, Mathe, Patel and Pikhurko [2015] proved that the ground set of the ordered probability spaces can be always $[0,1]$ with Lebesgue measure.

k-dimensional posets

A sequence of k-dimensional posets can be represented by k sequences of permutations.

This suggests an intuitive limit for k-dimensional poset sequences.

Question: When does such a sequence converge?

Question: What kind of limit we have?

Limit k-dimensional poset (or k-kernel)

A k-kernel $Z = (X_1, \dots, X_k)$ is a tuple of k uniform random variables X_1 to X_k in $[0, 1]$ (given by their joint distribution function).

Z-random poset $P(n, Z)$: Generate according to Z n points $Y^{(i)} = (X_1^{(i)}, \dots, X_k^{(i)})$ of $[0, 1]^k$, for $i = 1, \dots, n$.

$P(n, Z)$ is the poset $([n], \prec_P)$ such that $i \prec_P j$ if and only if $Y^{(i)} < Y^{(j)}$ (if and only if every coordinate of $Y^{(i)}$ is smaller than the corresponding coordinate of $Y^{(j)}$).

This model generalizes the random k-dimensional poset model (Graham Brightwell) (just take $X_1, \dots, X_k$ independently).

Convergent k-dim. poset sequences

Definition

A sequence of k-dimensional posets (B_n) is said to converge if, for every fixed poset P, the real sequence $(t(P, B_n))_{n \in \mathbb{N}}$ converges.

$t(P, B_n)$ is the probability that a random subposet of B_n has the same order of P.

Convergent k-dim. poset sequences

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A sequence of k-dimensional posets (B_n) is said to converge if, for every fixed poset F, the real sequence $(t(F, B_n))_{n\in\mathbb{N}}$ converges.

Theorem (Hoppen et.al, 2018)

For every convergent k-dimensional poset sequence (B_n), there exists a k-kernel $Z = (X_1, \ldots, X_k)$, such that, for every poset F,

$$\lim_{n\to\infty} t(F, B_n) \;=\; t(F, Z) \;:=\; \mathbb{P}\Big(P(m, Z) \cong F\Big),$$

where m is the size of F.

Convergent k-dim. poset sequences

Theorem (Hoppen et.al, 2018)

Let Z be a k-kernel. The sequence $(P(n, Z))_{n=1}^{\infty}$ converges to Z with probability one.

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For every convergent k-dimensional poset sequence (B_n), there exists a k-kernel $Z = (X_1, \ldots, X_k)$, such that, for every poset F,

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where m is the size of F.

Uniqueness of the limit ???

Definition

Let $Y(Z)$ be a random point in $[0,1]^k$ generated according to the k-kernel Z. The rectangular distance between k-kernels Z and Z':

$$d_{\square}(Z,Z') \;=\; \sup_{\Delta \in I[0,1]^k} \left| \mathbb{P}\big(Y(Z) \in \Delta\big) - \mathbb{P}\big(Y(Z') \in \Delta\big) \right|,$$

where $I[0,1]^k$ is the set of all k-dimensional intervals of $[0,1]^k$.

$$\delta_{\square}(B,Z') \;=\; \min_{realizer \;\; RofB} \; d_{\square}(Z_R,Z)$$

Uniqueness of the limit ???

Definition

Let $Y(Z)$ be a random point in $[0,1]^k$ generated according to the k-kernel Z. The rectangular distance between k-kernels Z and Z':

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where $I[0,1]^k$ is the set of all k-dimensional intervals of $[0,1]^k$.

$$\delta_\square(B, Z') = \min_{realizer\ \ RofB} d_\square(Z_R, Z)$$

Theorem (Hoppen et.al, 2018)

If $(B_n) \to Z$*, then* $\delta_\square(B_n, Z) \to 0$*.*

Parameter testing

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Parameter testing through subposets

Objective: accurately predict the value of a parameter $f(B)$ by looking at a randomly chosen subposet of much smaller size.

Definition

A parameter f is k-dim. testable if,
For every $\epsilon > 0$,
There exist positive integers $t < n_0$ s.t.:

If B is a k-dimensional poset of length $n > n_0$, then

$$\mathbb{P}\Big(|f(B) - f(sub(t, B))| > \varepsilon\Big) \leq \varepsilon.$$

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A parameter f is testable if it is k-dim testable, for every k.

Characterization of testable parameters

Theorem

A bounded poset parameter is k-dim. testable if and only if the sequence $(f(B_n))_{n\in\mathbb{N}}$ converges for every convergent sequence $(B_n)_{n\in\mathbb{N}}$ of k-dimensional posets.

A poset parameter f is bounded if there is a constant M such that $|f(B)| < M$ for every poset σ.

Immediate consequences

Testable Poset Parameters

- The subposet density $f_P(B) = t(P, B)$ for any fixed P.
- The density of pairs.

NOT Testable Poset Parameters (through subposets)

- The height (over n).
- The width (over n).