Combinatorial Games in Graphs: Pursuit, Coloring and Convexity

Rudini Sampaio

Universidade Federal do Ceará (UFC)
Departamento de Computação
PARGO Research Group
Fortaleza, Brazil

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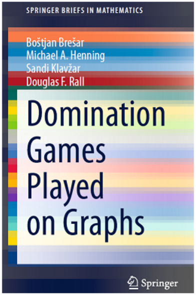
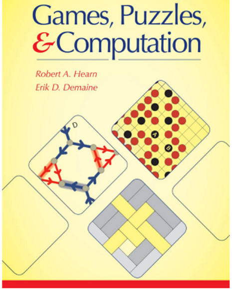
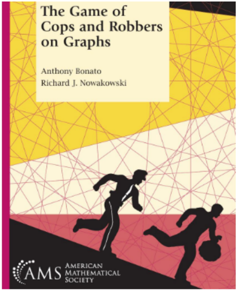
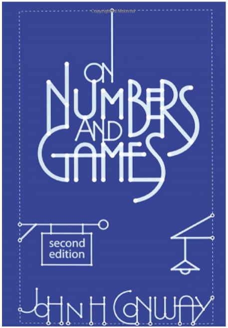
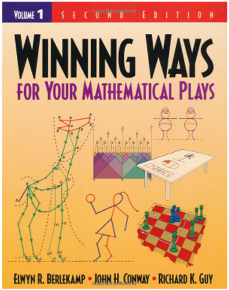
Combinatorial games

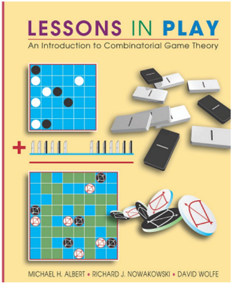












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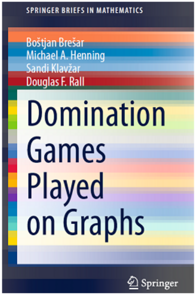

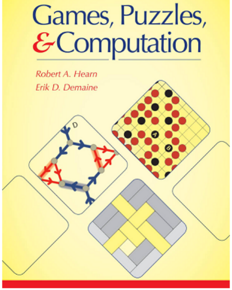

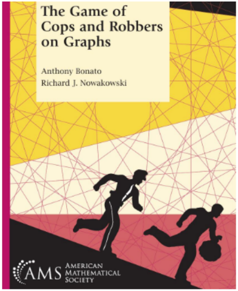

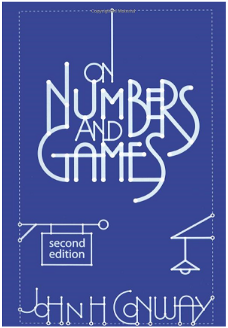

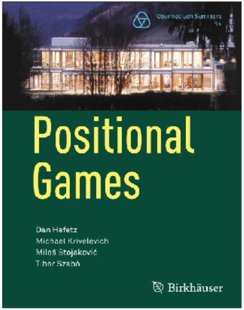

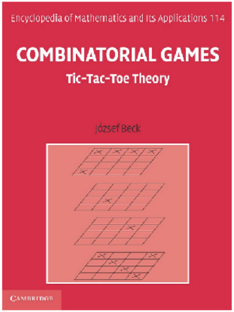

What is a Combinatorial Game?

- Two-person games with perfect information and no chance moves.
- Initial conditions: initial position + the first player.
- Players alternate moves.   Outcome: win, lose or tie (draw).
- Set of Terminal positions: from which no moves are possible.
- Finite games: rules to avoid loops (repetition of a position).

Variants of a Combinatorial Game

- **Normal variant:** the last to play wins (**achievement**).
- **Misère variant:** the last to play loses (**avoidance**).
- **Optimization variant:** A numerical parameter achieves some goal

Main question (decision problem)

- Zermelo-von Neumann Thm (1913): Given an instance of a game without draw, one player has a winning strategy.
- Does the 1st (first) player have a winning strategy?

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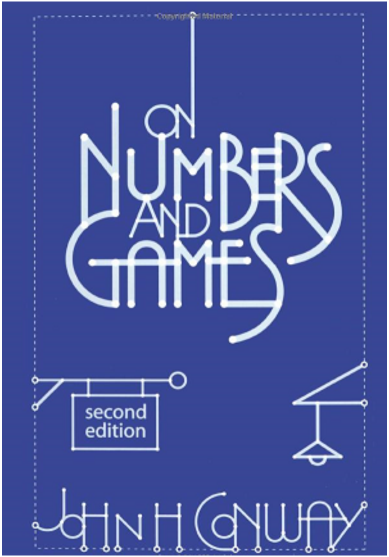
Classification of Combinatorial Games

- **Impartial:** both players have the same set of possible moves and same objectives at any turn. The only difference is the first to play.
- **Partizan:** non-impartial games.

Impartial Games can be represented by numbers



Impartial Games

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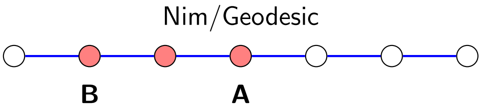

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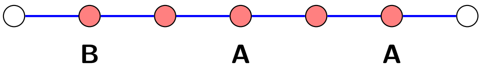
Nim/Geodesic



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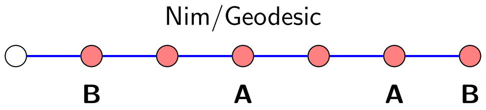

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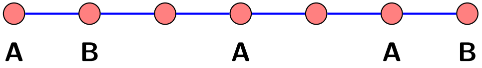

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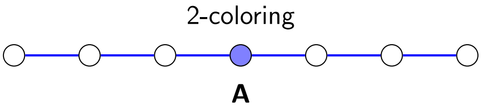

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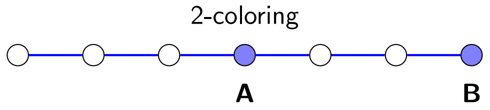

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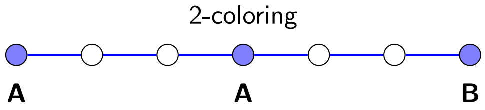

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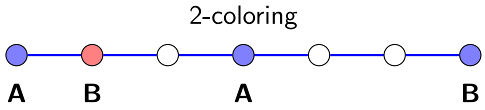

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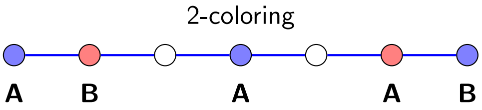

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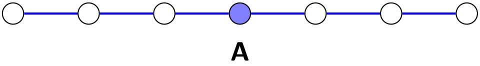
1-coloring / Kayles (Dudeney, 1908)



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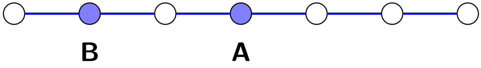

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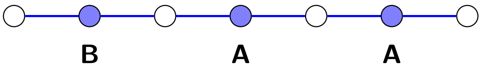

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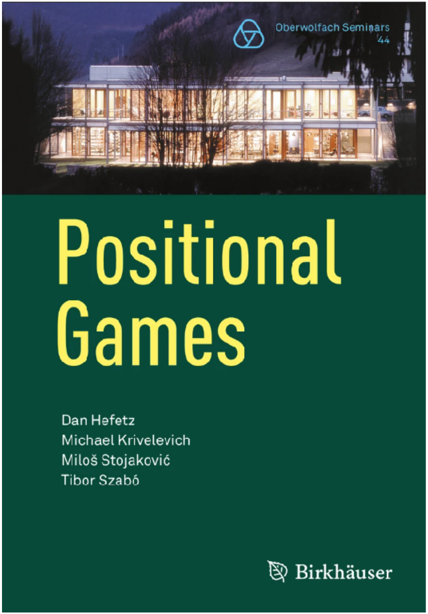
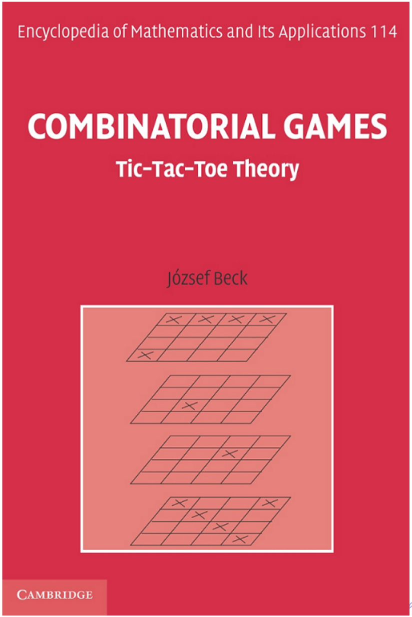
Positional Games: Maker-Breaker, Maker-Maker, ...





Positional Games: Tic tac toe, Hex, Maker Breaker, . . .

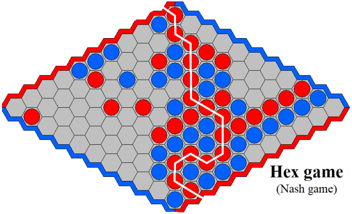
- Finite partizan combinatorial games with a finite set X (*the board) whose elements are called the positions,*
- *a family $\mathcal{F}$ of subsets of X - the winning (or losing) sets - and*
- *Conditions for winning.*
- *Players alternately claim unclaimed positions, until a player wins.*

Types of positional games

- Maker-Maker (strong positional): 1st to claim "winning set" wins. **Ex:** Tic tac toe, Hex.
- Avoider-Avoider: 1st to claim a "losing set" loses. **Ex:** Ramsey game (SIM): positions are edges of K_6 and "losing sets" are the triangles.
- Maker-Breaker: Maker wins by claiming a "winning set"; otherwise, Breaker wins. Draw is not possible. **Ex:** POSCNF game: CNF formula with non-negative literals. Maker wants to "make" a false clause. Breaker wants to satisfy the formula.
- Avoider-Forcer: Avoider loses by claiming a "losing set".

Positional Games: Tic tac toe, Hex, Maker Breaker, . . .

- **Strategy-Stealing existencial argument [John Nash, 1947]**: 1st player can force at least a draw in strong positional games.
- **Example (Hex):** 1st player wins Hex, a game invented by Piet Hein (1942), rediscovered by **John Nash (1947)** (*A Beautiful Mind*).
  Open problem: Find an explicit winning strategy for $n \geq 10$ in Hex.
- **Example (Chess):** It's not a positional game, and is not solved. However, white's advantage is a consensus.



Hex game
(Nash game)

- **[Reisch, 1981]:** Given Hex configuration, deciding if the 1st player wins is PSPACE-Complete.
- **[Rahman, Watson' 2023]:** 6-uniform Maker-Breaker and 7-uniform Maker-Maker are PSPACE-Complete.

Positional Games: Tic tac toe, Hex, Maker Breaker, ...

- **Strategy-Stealing existencial argument (John Nash, 1947)**: 1st player can force at least a draw in strong positional games.
- Van der Waerden Thm (1927): $\forall k, c : \exists n$: if $\{1, \ldots, n\}$ is c-colored, $\exists$ monochromatic arithm progr of size k. $W(k{\geq}3){=}[9, 35, 178, 1132, ?]$
  Maker-Breaker with $c = 2$ colors: $W^*(k{\geq}3){=}[5, 15, ?]$.
- Hales-Jewett Thm (1963): $\forall k, c :\ \exists n$: if n-dim k^n cube is c-colored, there is a monochromatic combinatorial line (rows, columns or main diagonals). Generalized Tic-tac-toe: $HJ^*(3) = HJ(3) = 3$. $HJ(4) = ?$
- Ramsey Thm (1930): $\forall k, c :\ \exists n$: if the edges of K_n are c-colored, there is a monochromatic K_k. $R(3) = 6$. Clique game: $R^*(3) = 5$.
- Erdős-Selfridge Thm (1973) for Maker-Breaker: Breaker wins if $\sum_{A \in \mathcal{F}} 2^{-|A|} < 1/2$. Or < 1 if Breaker is the first to play.

Recall the (main) types of positional games

- Strong positional: 1st to claim "winning set" wins. **Ex:** Tic tac toe.
- Maker-Breaker: Maker wants to claim a "winning set"; Breaker wants the opposite. Draw is not possible.

Coloring Games (two optimization variants)

ON THE COMPLEXITY OF SOME COLORING GAMES

Hans L. Bodlaender

November 1989

J Comb Optim (2013) 25:752–765

The game Grundy number of graphs

Frédéric Havet · Xuding Zhu

Coloring Games (two optimization variants)

Proper coloring

- The vertices of graph are colored
- Two adjacent vertices must receive distinct colors
- $\chi(G)$: chromatic number (min number in a proper coloring)

Greedy coloring

- Proper vertex coloring / colors are integers
- Take an ordering of the vertices.
- A vertex must receive the minimum available color.
- $\Gamma(G)$: Grundy number (max number in a greedy coloring)

$$\chi(G) \quad \leq \quad \Gamma(G)$$

Coloring Games (two optimization variants)

- **Instance:** a graph G and a set C of colors/integers
- Two players **Alice** and **Bob** alternate their turns in choosing an uncolored vertex to be **proper colored** by an integer of C
- Alice **starts** and **she wins** if all vertices are successfully colored; Otherwise, Bob wins the game
- Zermelo-von Neumann Th.: Alice or Bob has a winning strat

Graph coloring game ($\chi_g(G) \geq \chi(G)$)

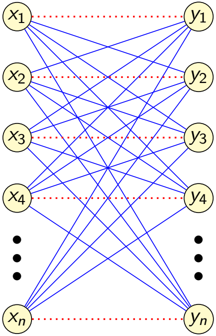
- Alice and Bob may use **any** possible integer of C
- **Game chromatic** number $\chi_g(G)$: minimum number of colors s.t. Alice has a winning strategy in the graph coloring game

Greedy coloring game ($\chi(G) \leq \Gamma_g(G) \leq \Gamma(G)$)

- Alice and Bob must use **the smallest** possible integer of C
- **Game Grundy number** $\Gamma_g(G)$: minimum number of colors s.t. Alice has a winning strategy in the greedy coloring game

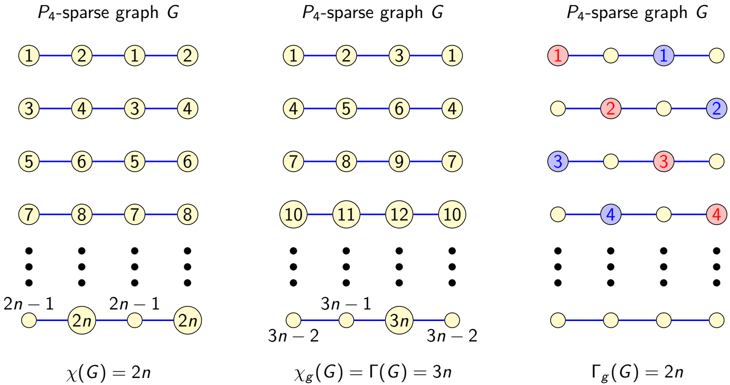
Coloring Games (example for both games)

- Complete bipartite graph without a matching
- **If Alice is the first** to play, Bob can force n colors: just play in the non-neighbor of Alice's last vertex.
- **If Bob is the first** to play, Alice wins with 2 colors.
  Coloring game: Alice colors the non-neighbor with the other color.
  Greedy game: color same side of Bob's first vertex.



Coloring Games (example for both games)

- P_4-sparse example: join of n P_4's



Coloring Games (known results)

- First considered by [Brams] and described by [Gardner'81, Math. Games column of Scientific American]
- Reinvented by [Bodlaender'91]: "*The complexity of the Color Construction Game is an interesting open problem*"
- trees ≤ 4 [Faigle...'93], outerplanar ≤ 7 [Kierstead...'94]
- $\chi_g \leq (\chi_a + 1)^2$ acyclic chromatic number χ_a [Dinski,Zhu'99]
- $\chi_g(P_k) \leq 3k + 2$ for partial k trees [Zhu'00]
- $\chi_g(G) \leq 5$ in cacti [Sidorowicz'07]
- Asympt. behavior $\chi_g(G(n,p))$ [Bohman, Frieze, Sudakov'08]
- Ex.value χ_g cartes prod K_2 w path/cycle/clique [Bartnick'08]
- Planar graphs: $\chi_g \leq 17$ [Zhu'08], $\chi_g \leq 13$ [Sekiguchi'14, girth≥ 4], $\chi_g \leq 5$ [Nakprasit'18, girth≥ 7]

- $\chi_g(F)$ poly trees no vertex deg 3 [Dunn et. al'15]: "*more than two decades later,* ***this question*** *remains open*".
- poly characterization game-perfect graphs [Andres,Lock'19]: "*the question of PSPACE-hardness remains open*".

Coloring Games (known results)

Greedy coloring game / Game Grundy number $\Gamma_g(G)$

- Introduced by [Havet, Zhu'13]
- $\Gamma_g(G) = \chi(G)$ in cographs [Havet,Zhu'13]
- $\Gamma_g(F) \leq 3$ in trees [Havet, Zhu'13]
- $\chi_g(G) \leq 7$ in partial 2-trees [Havet, Zhu'13]
- Two questions of [Havet,Zhu'13]
  - (*) $\chi_g(G)$ is upper bounded by a function of $\Gamma_g(G)$?
  - (**) $\Gamma_g(G) \leq \chi_g(G)$ for every graph G?
- (*) $=$ NO [Krawczyk,Walczak'15]
- (**) is still open

Coloring games (decision problem definition)

Zhu'99 open question: Graph coloring game "*exhibits some strange properties*". Does Alice have a winning strategy with $k + 1$ colors if she has one with k colors?

We define three decision problems for the graph coloring game:

- (Problem 1) Given G and k: $\chi_g(G) \leq k$?
- (Problem 2) Given G and k: Does Alice have a winning strategy with k colors?
- (Problem 3) Given G and $\chi(G)$: $\chi_g(G) = \chi(G)$?

Problems 1 and 2 are equivalent **iff** Zhu's question is true.
Problems 1 and 2 generalizations of Problem 3 - take $k = \chi(G)$

Problem 3 PSPACE-hard $\rightarrow$ Problems 1 and 2 PSPACE-hard

Reduce POSCNF $\rightarrow$ Problem 3: build G s.t. we know $\chi(G)$.

Coloring Games (our results)

- [Costa, Pessoa, **Sampaio,** Soares' 2020]: TCS.
  PSPACE-completeness of two graph coloring games.

Complexity results

- $\chi_g(G)$ is PSPACE-hard answer Bodlaender'91 open question
- $\Gamma_g(G)$ is PSPACE-hard
- Both decision problems are PSPACE-Complete

Exact/algorithmic results

- $\Gamma_g(G) = \chi(G)$ poly for split graphs
- $\Gamma_g(G) = \chi(G)$ poly for extended P_4-laden graphs, a class in the top of a hierarchy of graphs with few P_4's
- In both cases, Alice wins with $\chi(G)$ colors even if Bob can start the game and pass any turn

Coloring Games (variants: starting and passing turns)

Graph YZ-coloring game $g_{YZ}(\chi_g^{YZ}(G) \geq \chi(G))$

- $Y \in \{A, B\}$ and $Z \in \{A, B, no\ one\}$
- Y starts the game and Z may pass turns
- Alice and Bob may use **any** possible integer of C
- YZ-**game chromatic** number $\chi_g^{YZ}(G)$: min number of colors s.t. Alice has a winning strategy in the YZ-coloring game
- We omit Z when it is "no one": $\chi_g^A(G) = \chi_g(G)$ is the original game chromatic number.

Greedy YZ-coloring game gg_{YZ} $(\chi(G) \leq \Gamma_g^{YZ}(G) \leq \Gamma(G))$

- Same idea, but they must use **the min.** possible integer of C
- YZ-**game Grundy number** $\Gamma_g^{YZ}(G)$: min number of colors s.t. Alice has a winning strat in the greedy YZ-coloring game
- We omit Z when it is "no one": $\Gamma_g^A(G) = \Gamma_g(G)$ is the original game Grundy number.

[Andres,Lock'19]: Introduced the variants g_{YZ}. "*The question of PSPACE-hardness remains open for all these game variants*"

Coloring Games (connected variants)

Connected coloring game cg_Y ($\chi_{cg}^Y(G) \geq \chi(G)$)

- Similar to the graph coloring game: Alice starts and no one may pass turns. But colored vertices must induce a connected subgraph
- **Connected game chromatic** number $\chi_{cg}(G)$: min number of colors s.t. Alice has a winning strategy in the cg_A game.
- Introduced by [Charpentier,Hocquard,Sopena,Zhu'19]
- [CHSZ'19]: Alice wins with 2 colors in bipartite graphs
- [CHSZ'19]: Alice wins with 5 colors in outerplanar graphs
- [Bradshaw'20]: There are outerplanar 2-trees with $\chi_{cg}(G) = 5$

Connected greedy coloring game cgg_Y ($\Gamma_{cg}^Y(G) \geq \Gamma(G)$)

- Similar to the greedy coloring game: Alice starts and no one may pass turns. But colored vertices must induce a connected subgraph
- **Connected game Grundy** number $\Gamma_{cg}^Y(G)$: min number of colors s.t. Alice has a winning strategy in the cgg_Y game.

Coloring Game variants (our results)

- [Lima, Marcilon, Martins, **Sampaio**' 2022]: TCS. PSPACE-hardness of variants of the graph coloring game.
- [Lima, Marcilon, Martins, **Sampaio**' 2023]: TCS. The connected greedy coloring game.

Complexity results

- Game chromatic numbers χ_g^{YZ} are PSPACE-hard for all variants
- Game Grundy numbers Γ_g^{YZ} are PSPACE-hard for all variants
- Connected game chromatic numbers χ_{cg}^{Y} is PSPACE-hard
- Connected game Grundy numbers Γ_{cg}^{Y} is PSPACE-hard
- All the related decision problems are PSPACE-Complete, even if the number of colors is the chromatic number
- Polynomial algorithms for split graphs.

Coloring game: χ_g: Reduction from POS-CNF

CNF formula, only positive variables, Alice and Bob alternate turns setting variables true or false. Alice wins if the formula is true.

Example

$(X_1 \vee X_2) \quad \wedge \quad (X_1 \vee X_3) \quad \wedge \quad (X_2 \vee X_4) \quad \wedge \quad (X_3 \vee X_4)$.

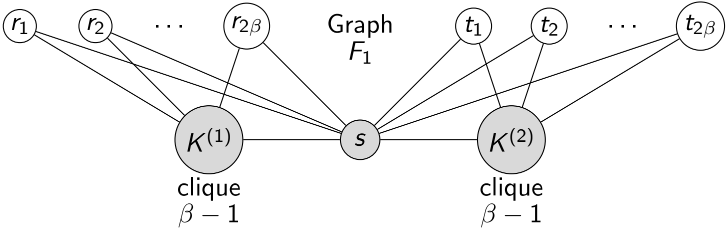
Bob has a winning strategy:

- X_1 True $\rightarrow$ X_4 False;
- X_2 True $\rightarrow$ X_3 False;
- X_4 True $\rightarrow$ X_1 False;
- X_3 True $\rightarrow$ X_2 False.

Good points

- POS-CNF is PSPACE-Complete
- If she/he has a winning strategy in POS-CNF, she/he also has a winning strategy if the opponent can pass turns.

Coloring game: χ_g: Important ingredient of the Reduction



Lemma:

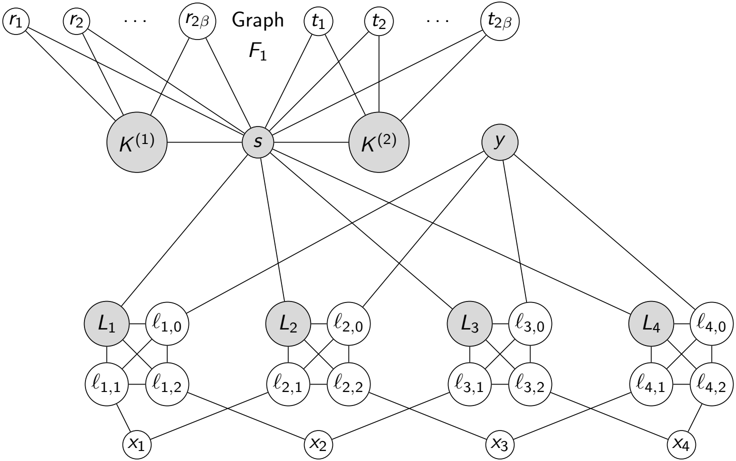
Alice has a winning strategy in F_1 with $2\beta - 1$ colors **iff** she colors vertex s first.

Proof:

If Alice does not color s first, Bob can color β vertices r_k/t_k, forcing 2β colors with clique $K^{(i)} \cup s$. If Alice colors s first, she can color $K^{(1)}$ and $K^{(2)}$ before Bob colors β vertices r_k/t_k.

Coloring game: χ_g: Reduction from POS-CNF

$$(X_1 \vee X_2) \quad \wedge \quad (X_1 \vee X_3) \quad \wedge \quad (X_2 \vee X_4) \quad \wedge \quad (X_3 \vee X_4).$$



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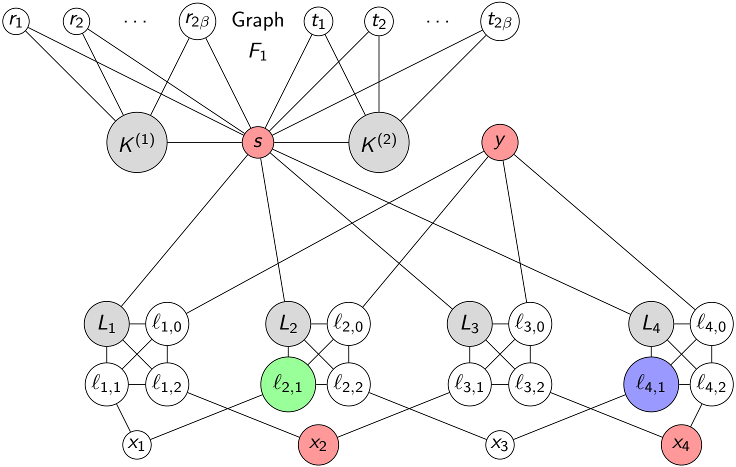


Coloring game: $\Gamma_g(G)$ is PSPACE-hard

Differently than the Graph Coloring Game, if Alice has a winning strategy with $k+1$ colors in the greedy coloring game she has one with k colors.

We define two decision problems for the greedy coloring game:

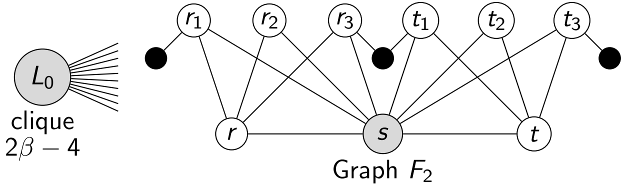
- (Problem 1') Given G and k: $\Gamma_g(G) \leq k$? That is: Does Alice have a winning strategy with k colors?
- (Problem 2') Given G and $\chi(G)$: $\Gamma_g(G) = \chi(G)$?

Problems 1' generalization of Problem 2' - take $k = \chi(G)$

Problem 2' PSPACE-hard $\rightarrow$ Problem 1' PSPACE-hard

Reduce POSCNF $\rightarrow$ Problem 2': build G s.t. we know $\chi(G)$.

Coloring game: Γ_g: Important ingredient of the Reduction



Lemma:

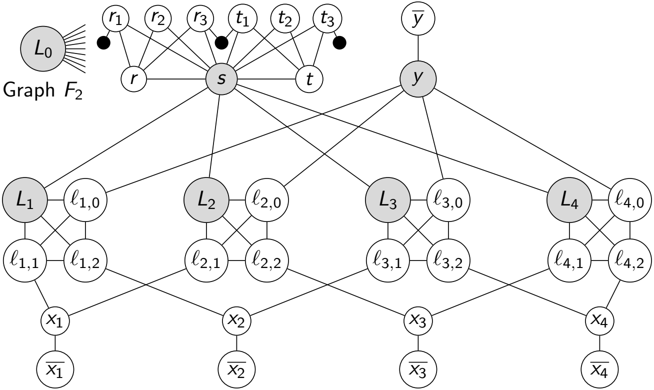
Alice has a winning strategy in F_2 with $2\beta - 1$ colors **iff** she colors vertex s first.

Proof:

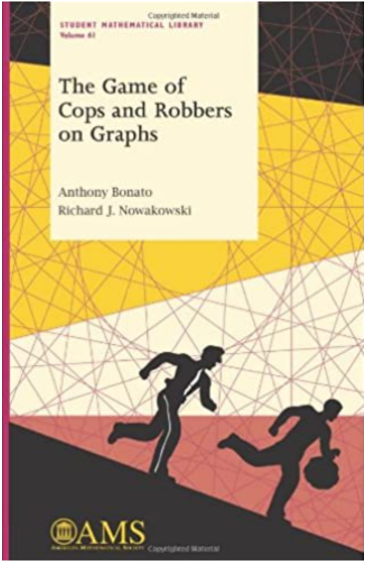
If Alice does not color s first (assume r wlg), Bob colors t_2 and a black vertex with 1, forcing 4 colors in a triangle $s - t - t_i$. If Alice colors s first, she can color r and t with 2 or 3 (black vertices will be 1), forcing 3 colors.

Coloring game: Γ_g: Reduction from POS-CNF

$$(X_1 \vee X_2) \quad \wedge \quad (X_1 \vee X_3) \quad \wedge \quad (X_2 \vee X_4) \quad \wedge \quad (X_3 \vee X_4).$$



Pursuit-evasion games: Cops and Robber



Game introduced in [Nowakowski, Winkler' 1983]

Pursuit-evasion games: Cops and Robber

- **Instance:** Graph G and an integer k.
- **The Game:** k cops (first) and 1 robber are placed at vertices of G.
- The cops (first) and the robber may move along an edge.
- The cops win if a cop occupies the same vertex of the robber.
- The robber wins if a configuration is repeated.
- **Cop number $c(G)$:** min k s.t. the cops have a winning strategy.

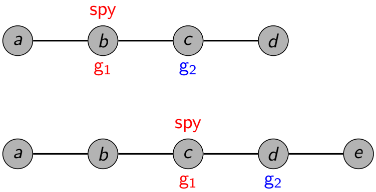
Literature

- Petersen's graph is the smallest graph with cop number =3
- [Kinnersley' 2015] Cops-and-Robber is EXPTIME-complete
- [Meyniel' 1985] Conjecture: $c(G) = O(\sqrt{n})$
- [Scott, Sudakov' 2011] $c(G) = O(n \ / \ 2^{(1-o(1))\cdot\sqrt{\log_2 n}})$
- [Prałat, Wormald' 2015] Meyniel holds a.a.s for $G(n, p)$
- [Aigner, Fromme' 1984] $c(G) \leq 3$ in planar graphs
- [Schröder' 2001] $c(G) \leq 1.5 \cdot g + 3$, where g is the genus
- [Bowler et al.' 2011] $c(G) \leq 1.333 \cdot g + 3.333$
- [Martins, **Sampaio**' 2018] $c(G) \leq 2$ in $(q, q\text{-}4)$-graphs with $n \geq q$

Pursuit-evasion games: Spy Game (s, d)

- [Cohen, Hilaire, Martins, Nisse, Pérennes' 2016]: Paths/cycles.
- [Cohen, McInerney, Nisse, Pérennes' 2020]: Polytime in trees.

- (s, d): spy speed $s \geq 2$ and surveillance distance $d \geq 0$
- **Instance:** Graph G and an integer k.
- **The game:** 1 spy (first) and k guards are placed on G. The spy may move along $\leq s$ edges and then each guard may move along 1 edge.
- **End:** Spy wins if she reaches a vertex at dist $> d$ from each guard after the guards' moves. The guards win if a game configuration is repeated.
- **Guard number $gn_{s,d}(G)$:** min k s.t. guards have winning strategy.



$gn_{2,0}(P_4) = 2, \quad gn_{2,1}(P_4) = 1$

$gn_{2,0}(P_5) = 3, \quad gn_{2,1}(P_5) = 1$

Spy Game: known results

- [Cohen, Hilaire, Martins, Nisse, Pérennes' 2016]: Paths/cycles.
- [Cohen, McInerney, Nisse, Pérennes' 2020]: Polytime in trees.

- [Cohen, Martins, McInerney, Nisse, Pérennes, **Sampaio**' 2018]: TCS *Spy-Game on graphs: Complexity and simple topologies.* NP-hard for any $s \geq 2$ and $d \geq 0$.
- [Costa, Martins, **Sampaio**' 2022]: TCS *Spy game: FPT-algorithm, hardness and graph products.* W[2]-hard in bipartite graphs for any $s \geq 2$ and $d \geq 0$. XP-algorithm on the number of guards.

Relation with other games

- Cops and Robber [Nowakowski, Winkler'83]: Similar for $s = 1$ if the guards are placed first in the Spy Game. Very different for $s \geq 2$.
- Eternal Domination [Goddard et al.'2005]: Spy Game with surveillance distance $s = \infty$ (or the diameter of the graph).

Spy Game: known results

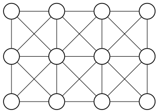
[Costa, Martins, **Sampaio**' 2022]: TCS
Spy game: FPT-algorithm, hardness and graph products.

- We begin the study of the spy game in grids and graph products.
- Strong product: $gn_{s,d}(G_1 \boxtimes G_2) \leq gn_{s,d}(G_1) \times gn_{s,d}(G_2)$.
- Strict upper bound: examples with King grids that match this upper bound and others for which the guard number is smaller.
- Cartesian/lexicographical products: this upper bound fails.
- Lexicographical products: Exact value for any distance $d \geq 2$.
- XP algorithm: The spy game decision problem is $O(n^{3k+2})$-time solvable for every speed $s \geq 2$ and distance $d \geq 0$.
- FPT algorithm: Spy Game is FPT on the P_4-fewness of the graph, solving a game on graphs for many graph classes.
- W[2]-hardness even in bipartite graphs when parameterized by the number k of guards, for every speed $s \geq 2$ and distance $d \geq 0$.

This hardness result generalizes the W[2]-hardness result of the spy game in general graphs [Cohen et al.'2018] and follows a similar (but significantly different) structure of the reduction from Set Cover in [2018]. However, the extension to bipartite graphs brings much more technical difficulties to the reduction, making this extension a relevant and non-trivial result.

Spy Game on the strong product (and King grids)

The strong product $G_1 \boxtimes G_2$ has vertex set $V(G_1) \times V(G_2)$ and vertices (u_1, u_2) and (v_1, v_2) are adjacent iff (a) $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$, or (b) $u_2 = v_2$ and $u_1 v_1 \in E(G_1)$, or (c) $u_1 v_1 \in E(G_1)$ and $u_2 v_2 \in E(G_2)$. The King grid $\mathcal{K}_{n,m}$ is the strong product of two path graphs $P_n \boxtimes P_m$.



King Grid $4 \times 3 \quad = \quad P_4 \boxtimes P_3$

Theorem 1: Let $s \geq 2$ and $d \geq 0$.

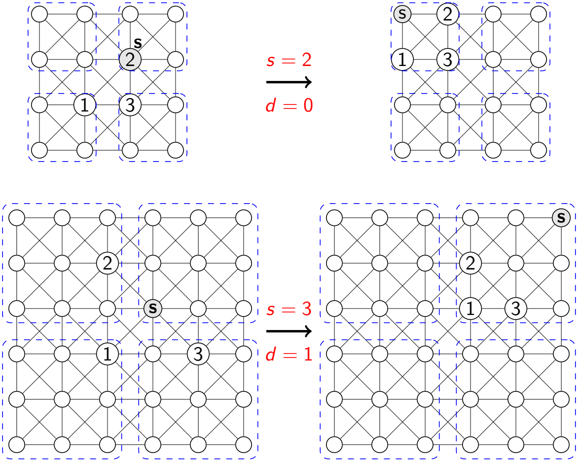
$$gn_{s,d}(G_1 \boxtimes G_2) \ \leq \ gn_{s,d}(G_1) \times gn_{s,d}(G_2).$$

Moreover, the equality holds if $gn_{s,d}(G_1) = 1$ or $gn_{s,d}(G_2) = 1$.

Proof sketch: Combining winning strategies in G_1 and G_2. For any guard h_1 in a vertex v_1 of G_1 and any guard h_2 in a vertex v_2 of G_2, consider a guard $h_1 h_2$ in the vertex (v_1, v_2) of $G_1 \boxtimes G_2$.

Spy Game: equality with $gn_{s,d}(G_1) = gn_{s,d}(G_2) = 2$

Lemma: For any $d \geq 0$, $s \geq d + 2$: $gn_{s,d}(P_{2d+4}) = 2$,
and $gn_{s,d}(P_{2d+4} \boxtimes P_{2d+4}) = 4$.

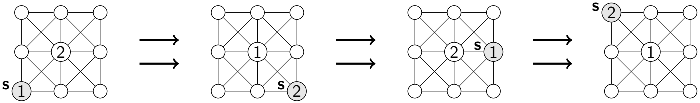


Spy Game: strict UB with $gn_{s,d}(G_1) = gn_{s,d}(G_2) = 2$

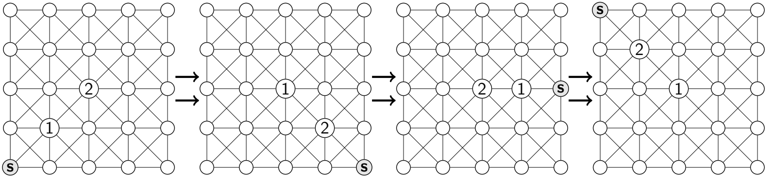
Lemma: For any $d \geq 0$, $s \geq 2d + 2$: $gn_{s,d}(P_{2d+3}) = 2$,

but $gn_{s,d}(P_{2d+3} \boxtimes P_{2d+3}) = 2$.

Example: surveillance distance $d = 0$ and spy speed $s \geq 2$



Example: surveillance distance $d = 1$ and spy speed $s \geq 4$



Spy Game: $gn_{s,d}(G_1)$ and $gn_{s,d}(G_2)$ greater than 2

Lemma: Let $d \geq 0$, $2 \leq k \leq 2d+2$ and $s \geq (k-1)(2d+3)$. Then $gn_{s,d}(P_{k(2d+3)}) = k+1$ and $k^2 \leq gn_{s,d}(P_{k(2d+3)} \boxtimes P_{k(2d+3)}) \leq (k+1)^2$.

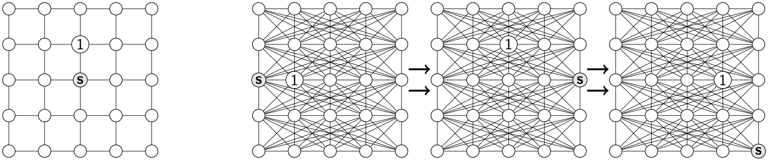
Proof: The vertex set of $P_{k(2d+3)} \boxtimes P_{k(2d+3)}$ can be partitioned into k^2 subsets of vertices which induces the King grid $P_{2d+3} \boxtimes P_{2d+3}$ each. If one of these subsets does not have a guard at some moment of the game, the spy can go to the vertex in the center of this subset and no guard can surveil the spy, which wins the game, since the distance from the center to the border is $d+1$ in the King grid $P_{2d+3} \boxtimes P_{2d+3}$. The spy can do this since her speed is at least $(k-1)(2d+3)$ (notice that the diameter is $k(2d+3)-1$ and the maximum distance between two centers of this subsets inducing the King grid $P_{2d+3} \boxtimes P_{2d+3}$ is at most $k(2d+3)-1-2(d+1) = (k-1)(2d+3)$). Thus, at least k^2 guards are necessary. Moreover, in [Cohen et al.'18], the exact value of $gn_{s,d}(P_n)$ was determined for any triple $(s \geq 2, d \geq 0, n \geq 2)$:

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d+2+\left\lfloor \frac{2d}{s-1} \right\rfloor} \right\rceil$$

With this, we have that $gn_{s,d}(P_{k(2d+3)}) = k+1$ for any $d \geq 0$, $2 \leq k \leq 2d+2$ and $s \geq (k-1)(2d+3) > 2d+1$. Then, from the upper bound, $(k+1)^2$ guards are sufficient and we are done. □

Spy Game: Cartesian and Lexicographical products

Lemma: $gn_{2,1}(P_5 \square P_5) > gn_{2,1}(P_5)^2 = 1$
$gn_{2,1}(P_5 \cdot P_5) > gn_{2,1}(P_5)^2 = 1$



Spy Game: Lexicographical product: exact value for $d \geq 2$

Theorem: Let $s \geq 2$, $d \geq 2$ and let G_1 and G_2 be two graphs.
If G_1 has no isolated vertex, then:

$$gn_{s,d}(G_1 \cdot G_2) \quad = \quad gn_{s,d}(G_1).$$

Otherwise:

$$gn_{s,d}(G_1 \cdot G_2) \quad = \quad \max \{ \, gn_{s,d}(G_1), \; gn_{s,d}(G_2) \, \}.$$

Spy Game: XP-algorithm on the number of guards

Theorem: $k \geq 1$, $s \geq 2$ and $d \geq 0$. It is possible to decide in XP time $O(n^{3k+2})$ if the spy has a winning strategy against k guards in the (s, d)-spy game on G.

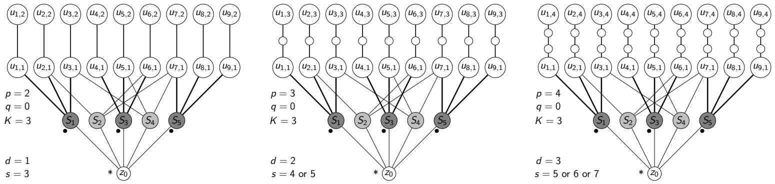
Proof: $2n^{k+1}$ game configurations.
Spy config: the spy is the next to move (spy/guards placed at vertices).
Guard config: the guards are the next to move.
Spy winning configuration: spy at distance $> d$ from any guard.
Mark all spy winning configurations. **Repeat** the following until no more configurations are marked. Mark the guard configurations that only lead to marked spy configurations (in words, any guards' move will lead to a spy winning configuration). Mark the spy configurations that lead to at least one marked guard configuration (in words, there is a spy's move which leads to a spy winning configuration).

If there is a vertex u such that any spy config with the spy in u is marked, then the spy has a winning strategy (by occupying vertex u first). Otherwise, the guards have a winning strategy. □

Spy Game: W[2]-hardness in bipartite graphs

- **Case 1:** $s > 2d + 2$
- **Case 2:** $s = 2d + 2$
- **Case 3:** $d + 1 < s < 2d + 2$
- **Case 4:** $s \leq d + 1$ and $s < 2(r + 1)$
- **Case 5:** $s \leq d + 1$ and $s = 2(r + 1)$
- **Case 6:** $s \leq d + 1$ and $s > 2(r + 1)$

where $r = d \mod (s - 1)$ is the remainder of the division of d by $s - 1$.



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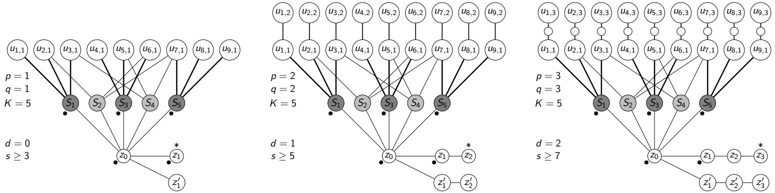

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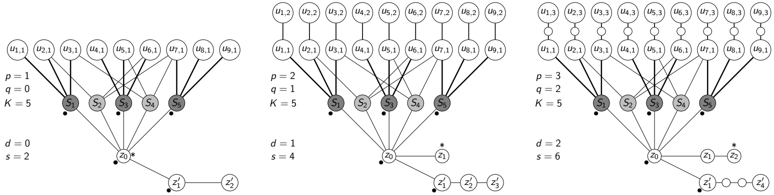

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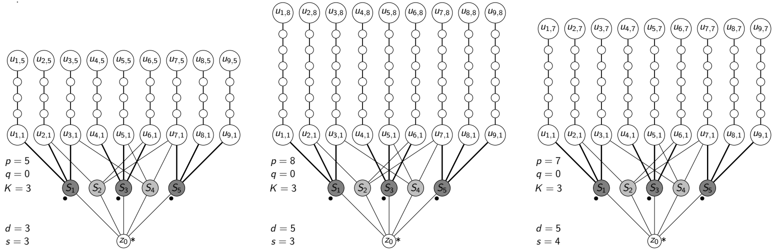

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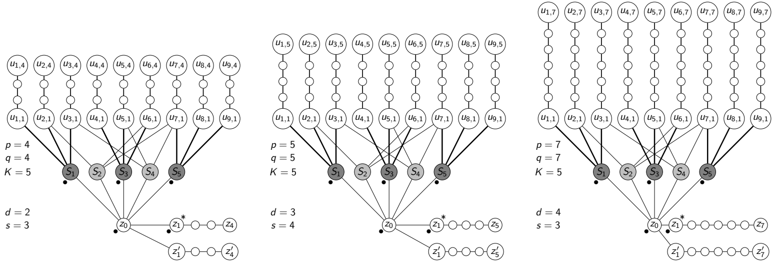
- **Case 5:** $s \leq d + 1$ and $s = 2(r + 1)$
- **Case 6:** $s \leq d + 1$ and $s > 2(r + 1)$

where $r = d \mod (s - 1)$ is the remainder of the division of d by $s - 1$.



Spy Game: W[2]-hardness in bipartite graphs

- **Case 1:** $s > 2d + 2$
- **Case 2:** $s = 2d + 2$
- **Case 3:** $d + 1 < s < 2d + 2$
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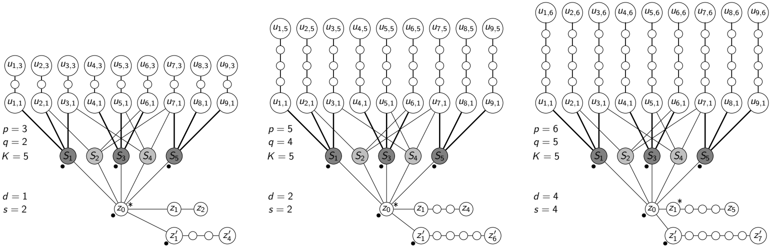


Convexity Games

Annals of Discrete Mathematics 20 (1984) 323
North-Holland

CONVEXITY IN GRAPHS: ACHIEVEMENT AND AVOIDANCE GAMES

Frank HARARY

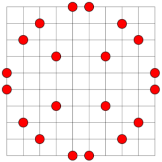
University of Michigan, Ann Arbor, Michigan, U.S.A.

Reference

[1] F. Harary and J. Nieminen, Convexity in graphs, J. Differential Geometry 16 (1981) 185–190.

Convexity games: The General Position Game

- Points in general position: no three points in the same line.



Maximum General Position Set in the 10 x 10 grid:

20 points (2 per line and 2 per column)

- Dudeney's "Puzzle with Pawns" in the book "Amusements in Mathematics" of 1917.
- "No-Three-In-Line Problem": Given n, find the maximum number of points in general position in the $n \times n$ grid.
- CS variation: given points in the plane $n \times n$, find the maximum number of points in general position.
- Graph variation: given a graph, select the maximum number of vertices in general position: no selected vertex is in the shortest path between other two selected vertices.

GP-game: General Position versus Geodesic Convexity

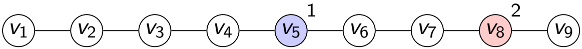
- Geodesic closure $I[S]$ of S: S and all vertices in shortest paths between two vertices of S.
- Equivalent to the Interval Set of the geodesic convexity.

- General Position Problem: given a graph G, select the max number of vertices in general position: No selected vertex is in a geodesic between other two selected vertices
- Max Geodesic Convex Set: given G, select the maximum number of vertices such that: No **non**-selected vertex is in a geodesic between two selected vertices

- Classical problems: There are many papers investigating them

General Position Game Variants

- Buckley-Harary-1985' geodesic game:
  two players (A/B) alternately select vertices which are not in the geodesic closure of the vertices selected so far.



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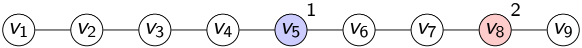

- **Normal variant:** The last to play wins (achievement game)
- **Misèrè variant:** The last to play loses (avoidance game)

- Zermelo'1913: One of the players has a winning strategy
- Problem: Given a graph, Player A has a winning strategy?

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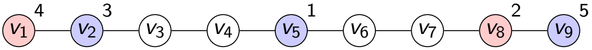

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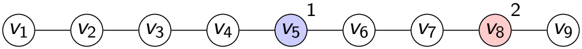

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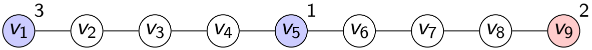

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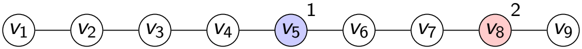

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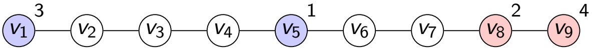

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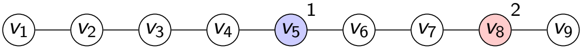

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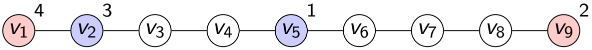

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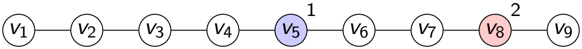

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Geodesic games: Known results

Geodesic games

- [Buckley-Harary-1985]: Solved for some graph classes.
- [Nečásková-1988]: Solved for wheel graphs.
- [Haynes-Henning-Tiller-2003]: Trees and complete multipartite.

General Position games

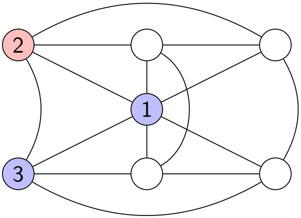
- [Chandran-Klavžar-Neethu-2021]: achievement game on Hamming graphs, Cartesian and lexicographic products.

General Position Game: Our results

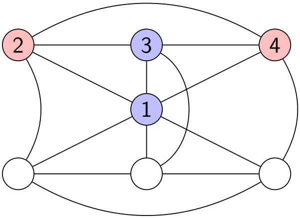
[Chandran, Klavzar, Neethu, **Sampaio**' 2022]: TCS. *The general position avoidance game and hardness of general position games.*

- We prove that the 2 games are PSPACE-Complete even in graphs with diameter at most 4:
  - 2021' achievement general position game
  - 2021' avoidance general position game
- For this, we had to prove that the misèrè versions of the 1978'Node-Kayles game and the 1978'Clique-Forming game are PSPACE-Complete: games of obtaining an indep. set or a clique, resp., where the loser is the last to pick a vertex.
- Polytime algorithms of gp-avoidance game in rook's graphs, grids, cylinders, and lexicographic products with complete second factors.

GP game: Examples



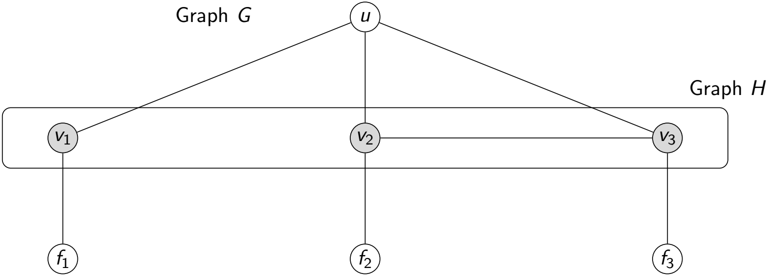
Player A wins the normal gp and clique-forming games



Player A wins the misèrè gp and clique-forming games

GP game: normal is PSPACE-Complete

Reduction from the Clique-Forming game: diameter 4 graph.



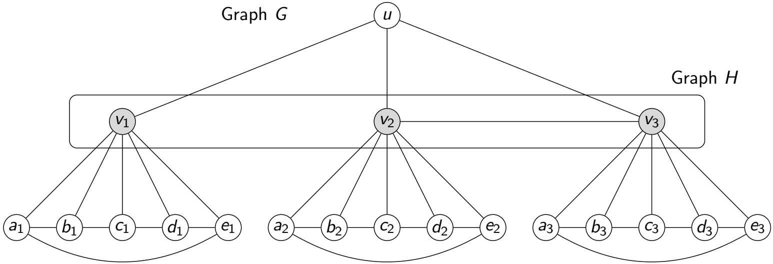
Player A has a winning strategy in the gp-achievement game

if and only if

Player B has a winning strategy in the Clique-forming game

GP game: misère is PSPACE-Complete

Reduction from the Clique-Forming game: diameter 4 graph.



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Hull Game: Our last results

[Araújo, Folz, Freitas, **Sampaio**' 2023]: LAGOS. *Complexity and winning strategies of graph convexity games.*

- We prove that the hull game is PSPACE-Complete even in graphs with diameter 2 in the normal and the misère variants:
  - 1984' hull game
- Reduction from the normal and misèrè variants of the 1978'Node-Kayles game and the 1978'Clique-Forming game, which are PSPACE-Complete: games of obtaining an indep. set or a clique, resp., where the loser is the last to pick a vertex.
- Polytime algorithms of convex geometries.
  - geodesic convexity and Ptolemaic graphs
  - monophonic convexity and chordal graphs
  - etc...

The END

Questions ??