Even Pairs in Planar Perfect Graphs

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Abstract

An even pair in a graph is a pair of non-adjacent vertices such that every induced path between them has even parity. It is known that finding an even pair in a general graph is an NP-hard problem [1]. In this paper, we give a proof, for planar perfect graphs, of the following conjecture due to Bruce Reed [3]: there is polynomial time algorithm to decide if a perfect graph has or not an even pair.

Key words: even pairs, odd pairs, perfect graphs, planar graphs, decomposition algorithms, graph colouring, NP-completeness.

1 Introduction

An *even pair* (resp. *odd pair*) in a graph is a pair of non-adjacent vertices such that every induced path between them has even (resp. odd) parity. Even pairs have been used to develop efficient combinatorial algorithms to color optimally some classes of perfect graphs, and to prove that certain classes of graphs are perfect [4]. In addition, they are related to the Strong Perfect Graph Conjecture [5]. In 1990, Bienstock proved that to decide whether a general graph has or not an even pair is a co-NP-complete problem [1]. In 1991, Bruce Reed conjectured that to decide if a perfect graph has an even pair is polynomially time solvable [3].

The perfectly orderable graphs can be cited as an example of the hardness of Reed's conjecture. If G is perfectly orderable, then G and every induced subgraph H of G has an even pair, however no one has being able to give a polynomial algorithm to finding it.

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Reed's conjecture has been confirmed for some classes of perfect graphs. For example, Meyniel graphs [7] and weakly triangulated graphs [8]. In this paper, we give a proof of Reed's conjecture restricted to planar perfect graphs, that is, we prove the following theorem:

Theorem 1 *Deciding whether a planar perfect graph has or not an even pair is polynomially time solvable.*

A first proof of this theorem was found by W-L. Hsu and O. Porto in 1993, which, however, have never presented it. Our polynomial time algorithm is strongly based in decomposition for planar perfect graphs obtained by W-L. Hsu in 1987 [2].

2 Preliminaries on Hsu's decomposition

Given a planar graph $G = (V, E)$, Hsu [2] associates with G a decomposition tree T whose root is G. If a graph H is a node of T and admits a cutset Q of one of certain types, then the children of H in T are obtained from the components of $H - Q$ by adding some new A-*artificial vertices.* If H has no such cutsets, then H is a leaf of T. Moreover, a cutset Q is applied to a node H only if none of the children of H are isomorphic to H. Hsu showed that G is perfect if and only if every leaf of T belongs to the union of three special classes, namely S, L and C [2]. Our algorithm also decomposes the graph G in a very similar way. Below, we briefly describe the types of cutsets which are relevant in our algorithm. Let Q be a cutset of H, and $B_1, \ldots, B_k$ the components of $H - Q$.

- *Types II-a and III-a: Q is a clique of size 2 and 3, respectively,* and the children are the graphs $H[V(B_i) \cup Q]$ for each $i = 1, \ldots, k$.
- *Type II-c: $Q = \{a, b\}$ with a, b not adjacent and there is a chordless odd $\{a, b\}$-path in H.* Then the i-th child is defined by taking $H[V(B_i) \cup Q]$ and adding a chordless $\{a, b\}$-path of length 3. The interior vertices of this path are A-artificial.
- *Types I, II-b, III-b and IV: cutsets with 1, 2, 3 and 4 vertices, respectively.* The occurrence of one of these types implies the existence of an even pair in G [2].

The decomposition along a cutset of a given type is applied only if there is no possible decomposition along cutsets of lower types. The Class C consists of (some) planar comparability graphs. The Class L consists of planar line-graphs of bipartite graphs such that every vertex belongs to exactly two cliques of length at most 3. Finally, the Class S consists of a finite family of finite graphs, whose membership is easy to test.

3 Even pairs in planar perfect graphs

Let T be the tree of Hsu's decomposition of a planar perfect graph G. The following lemma deals with the easy cases where an even pair can be found along Hsu's decomposition:

Lemma 2 *If a pair of non-artificial vertices forms an even pair in a leaf of T or if cutsets of type I, II-b, III-b or IV appear along Hsu's decomposition, then G has an even pair. Moreover, if some leaf of T, non-isomorphic to K_3 or K_4, belongs to $C \cup S$, then G has an even pair that is easily determined.*

Suppose now that T has only cutsets of types II-a, II-c or III-a and its leaves belong to class L or are isomorphic to a K_3 or a K_4. The following lemmas show us how we can modify T to find an even pair in G.

Let H be a node of T that has a cutset Q of one of types II-a, II-c or III-a. Suppose that H has an even pair $\{x, y\}$ which arise in different children H_1 and H_2 of H, respectively, while neither H_1 or H_2 has an even pair.

Lemma 3 *Suppose that Q is of type II-a or III-a. Then all the induced paths from x (resp. y) to one vertex of Q avoiding the others are odd.*

Lemma 4 *Suppose that $Q = \{a, b\}$ is of type II-c. Then all the induced paths from x (resp. y) to a avoiding b and all the induced paths from x (resp. y) to b avoiding a have the same parity, even or odd, and $H - Q$ has exactly two components.*

As a consequence of the above lemmas, we introduce new *Z-artificial* vertices in the children obtained by cutsets of Type II-a, II-c and III-a, as follows:

- *Type II-a and III-a: Q is a clique.* In this case the children are the graphs $H[V(B_i) \cup Q] + z$ for each $i = 1, \ldots, k$, with edges from z to every vertex in Q. If some child has a vertex, different from z, adjacent to every vertex in Q, we mark z, in this child, as a *diamond* Z-artificial vertex. Let z be colored *white.*
- *Type II-c: $Q = \{a, b\}$ with a, b not adjacent and a and b form an odd pair.* Then the i-th child is defined by taking $H[V(B_i) \cup Q]$, adding between a and b a chordless path aa_1a_2b of length three, whose interior A-artificial vertices are a_1 and a_2, and adding a Z-artificial vertex z adjacent to a and a_1. If $H - Q$ has exactly two components, let z be colored *blue.*

The addition of Z-artificial vertices, with respect to the previous lemma, will cause that $\{x, z\}$ (resp. $\{y, z\}$) will form an even or an odd pair in H_1 (resp. H_2). Our main idea is, after detecting these two even or odd pairs in H_1 and H_2, to match them along z in order to detect that $\{x, y\}$ forms an even pair of H. Observe that we can iterate this procedure. As a consequence, if G has an even pair, then there exists a sequence of matchable pairs $\{x, z_1\}$, $\{z_1, z_2\}$, $\ldots$, $\{z_k, y\}$, in different leaves of T, meaning that $\{x, y\}$ is an even pair of G. However, finding such sequence does not mean that $\{x, y\}$ forms an even pair in G. The following lemmas overcome this difficulty. Let us denote by Q_i the cutset that introduced z_i in a sequence of matchable pairs.

Lemma 5 *Let G be a planar perfect graph containing an even pair. Then G contains a pair $\{x, y\}$ such that there exists a sequence of matchable pairs $\{x, z_1\}, \ldots, \{z_k, y\}$ in the leaves of T, satisfying the following: (i) $Q_{i(i=2,\ldots,k-1)}$ is of Type II-c, and $G - Q_i$ has exactly two components; (ii) the number of odd pairs in the sequence is even.*

Lemma 6 *Let G be a planar perfect graph containing a sequence of matchable pairs $\{x, z_1\}$, $\ldots$, $\{z_k, y\}$ in the leaves of T satisfying the following: (i) $Q_{i(i=2,\ldots,k-1)}$ is of Type II-c with $G - Q_i$ having exactly two components; (ii) the number of odd pairs in this sequence is even. Then G has an even pair which is easily determined.*

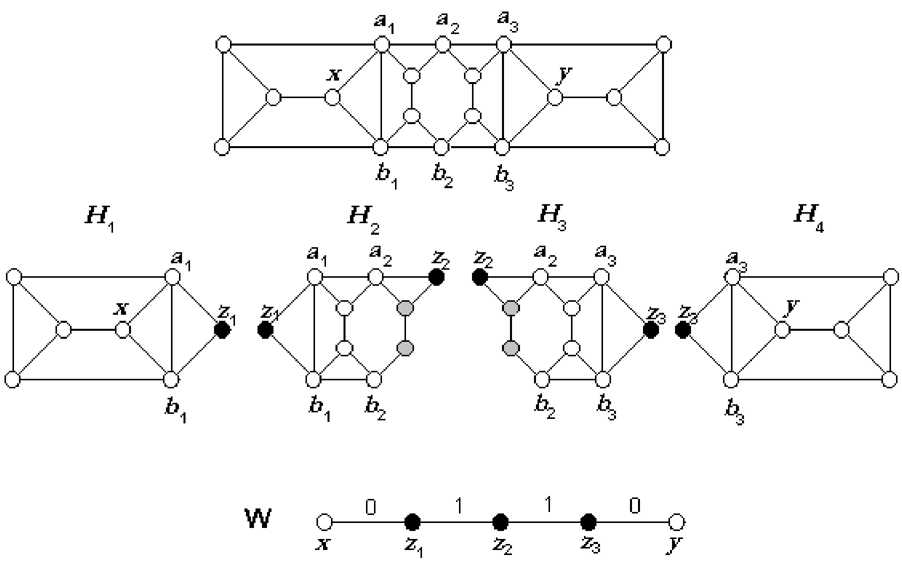

Fig. 1. Recuperation of the only even pair $\{x,y\}$ of H. The Z-vertices are in black, and the A-vertices are in gray

As an example, in Figure 1, we can match $\{x,z_1\}$, $\{z_1,z_2\}$, $\{z_2,z_3\}$ and $\{z_3,y\}$ as even, odd, odd and even pairs, respectively. Note that the number of odd pairs is even, and that the cutset $\{a_2, b_2\}$ is of type II-c.

Now, the problem is reduced to finding even and odd pairs in a special class of LGB graphs. W-L. Hsu, in Theorem 9.2 of [2], characterized the even and odd pairs in a graph belonging to L. S. Hougardy did it for general LGB graphs [6]. Both characterizations imply a polynomial time algorithm to determine those even pairs that are used in the algorithm below:

(1) Let G be a perfect planar graph and let T be its modified Hsu's decomposition tree. Let Z be the set of the Z-artificial vertices and W an empty queue.
(2) If a cutset of type I, II-b, III-b or IV is used in step (1), or if there is a leaf H of T such that $H - Z$ belongs to $S \cup C$, or $H - Z$ has an even pair of non-artificial vertices, then return *"G has an even pair easily determined"*.
(3) For each leaf H of T, let Z_D be the set of diamond Z-artificial vertices of H. Let us consider $H - Z_D$ and put in W the even and odd pairs whose artificial vertices are colored. Finally, put in W the even pairs with at least one of the vertices belonging to Z_D.
(4) If a pair of matchable non-artificial vertices satisfying the requirements of Lemma 6 is found in W, return *"G has an even pair easily determined"*. Otherwise, return *"G has no even pairs"*.

Step (1) takes $O(n^3)$ time [2]. If an even pair was not detected along the decomposition, we examine its leaves, applying Hsu's or Hourgady's procedures to find even and odd pairs in LGB graphs. This can also be done in $O(n^3)$. Finally, by building an auxiliary graph from W, it is also possible to find the pairs of matchable properly colored non-artificial vertices and determine whether or not G contains an even pair in $O(n^3)$.

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